TITLE:

Studies on Wave Propagation in Random Media and the Related Subjects(Dissertation_全文)

AUTHOR(S):

Ogawa, Shigeyoshi

CITATION:

Ogawa, Shigeyoshi. Studies on Wave Propagation in Random Media and the Related Subjects. 京都大学, 1975, 工学博士

ISSUE DATE:

1975-09-23

URL:

https://doi.org/10.14989/doctor.k1658

RIGHT:

京都大学学術情報リポジトリ KURENAI 紅
Kyoto University Research Information Repository

STUDIES ON WAVE PROPAGATION IN RANDOM MEDIA AND THE RELATED SUBJECTS

SHIGEYOSHI OGAWA

MARCH 1975

I
315
1-0

STUDIES ON WAVE PROPAGATION

IN

RANDOM MEDIA AND THE RELATED SUBJECTS

SHIGEYOSHI OGAWA

MARCH 1975

Studies on Wave Propagation

in

Random Media and the Related Subjects

by

Shigeyoshi OGAWA

Submitted in partial fulfillment of the

requirement for the degree of

DOCTOR OF ENGINEERING

(Applied Mathematics and Physics)

at

KYOTO UNIVERSITY

Kyoto, Japan

March 1975

Preface

As mathematical models of such physical phenomena that have some random characters in their features, problems on partial differential equations with random coefficients have been studied by numerous physicists and mathematicians in recent years. Especially the equations of hyperbolic type have been extensively studied in connection with problems of wave propagation in random media. Studies on this subject have been mainly concerned with the equations whose coefficients are assumed to be smooth, or to have other nice properties from the viewpoint of mathematics. However there are problems in physics ,or in engineerings where we can not expect those nice properties on the coefficients. For example, it is customary for the engineers to think of the "white noise" as a random disturbance which enters into the observations of signals or some physical quantities. Also in physics, there are problems where we must take account of the effects of the Brownian motion as the results of the thermodynamical phenomena. A part of these phenomena may be represented by partial differential equations with the white noise as their coefficients.

The purpose of this dissertation is to study the initial value problems of partial differential equations of the first order with the white noise as their coefficients, as interesting but extraordinary cases of those in the theory of "wave propagation in random media". Our interests will be restricted to the problems of the following two typical equations ;

(I), A partial differettial equation which has the white noise as an external random force term.

(II), A partial differential equation which has a "diffusion process" as its characteristic line.

As a preparation for studying these subjects, we shall introduce the new types of stochastic integrals and will investigate their properties in some detailes.

The author wishes that this dissertation will contribute to the theory of wave propagation in random media as a mile-stone.

Shigeyoshi OGAWA

Kyoto, Japan

March 1975.

ACKNOWLEDGMENT

For many of this dissertation a great debt is owed to Professors Hisashi Mine and Masaya Yamaguti of Kyoto University. The author would like to express his sincere appreciation to Professors H. Mine and M. Yamaguti for their continual encouragements and numerous helpful suggestions.

The author also wishes to express his gratitude to Associate Professor Toshiharu Hasegawa for his invaluable encouragements.

The author has to express his thanks to Dr. Takaaki Nishida with whom the author had many helpful discussions while he was a student.

CONTENTS

CHAPTER I INTRODUCTION

1. 1, Problems in the Theory of Wave Propagation in Random Media.

The theory of wave propagation in random media which is concerned with the wave motions in random media, has its origin in practical problems of radio wave propagation through the atomosphere and ionosphere, sound wave propagation in the ocean, light transmission through the atomosphere, etc.. It is a rather fresh field of applied mathematics and physics where the most of work has been done only in the last two decades by numerous mathematicians and physicists, such as L. A. Chernov ([3]), U. Frisch ([9],[10]), J. B. Keller ([24],[25]), V. I. Tatarski ([40]) and so on, (we can obtain an almost complete survey and references on this subject in U. Frisch [10] or in L. A. Chernov [3]). From the viewpoint of physics, problems in the theory are devided into the following two typical cases ; (i) Problems on wave propagation in continuous random media such as the turbulent fluids. (ii) Problems on scattering of waves by randomly distributed scatterers, (cf., Silver [36]). In this dissertation, we are mainly concerned with the problems in the former case.

In general, the wave motion in a continuous medium can be described as a function of space and time which satisfies a linear partial differential equation of hyperbolic type with variable coefficients, as a consequence of the physical laws governing the wave motion. Thus in mathematical formulations, problems on wave propagation in continuous random media are reduced to those on linear partial differential equations with random functions as their coefficients. Since a partial differential equation with random coefficients is no more than a family of nonrandom differential equations depending on a random parameter which assignes to each equation its frequency of realization, it may seem that the theory is properly included in the theory of nonrandom partial differential equations with variable coefficients. However, as was pointed out by many authors (U. Frisch [10], J. B. Keller [24]), we can expect to find out more about random problem than that found merely from individual nonrandom problems, because the final purpose of the random theory is essentially different from that of nonrandom theory. To explain this is to explain the reason why did the randomness be introduced into the problems. The reasons are quite the same as those for other theories of random phenomena; (i) In practical situations, we can not expect to have the complete knowledges about the physical quantities in the phenomena, except the statistical characters of them. Therefore the problems must be analyzed in a statistical manner. (ii) Even though we can get the complete knowledges on them, it may be of no worth or impractical to solve the reduced random equations, since in most cases (e.g., wave propagation in a turbulent medium) the coefficients of equations may be too complicated to apply the nonrandom theory of partial differential equations. In other words, the final purpose of the random theory is in the determination of statistical characters of random wave motions and not in a precise determination of wave motions.

The determination of statistical characters of a random wave motion is difficult in most cases, because the wave motion depends nonlinearly on the coefficients that bring the randomness into the equation. In order to get the equation which a specified moment of the random wave must satisfy, the various approximation technics were introduced for this aim. These technics are not rigorous in a mathematical sense but produce the results that compare well with the experiments and are called " dishonest methods ", or " nearly dishonest methods " following the terminologies of J. B. Keller and U. Frisch. (The " honest method " is a direct method in which first we solves the individual nonrandom equations and determines its statistical quantities at the second step.) These dishonest methods are the followings ; The diagram method which was introduced by Bourret [1], and extended by U. Frisch [11], the random Taylor expansions, the method of smoothing which was introduced by Primas [35] and Tatarski-Gertsenshtein [41], etc.. Nowadays, it is understood that one of the most important problems in the theory is to justify these approximation technics or is to introduce a more convenient technic which will cover wider regions than these technics.

We have given a short sketch of the theory from the viewpoint of applied mathematics. The random theory is also a fruitful and interesting one from the viewpoint of physics. In fact there are phenomena that can be understood well through the random theory such as the energy transfer between the different wave modes of a random wave (K. Hasselmann [14]), the stochastic acceleration in plasma physics (P. A. Sturrock [39]), etc.. Since the theory is based on the mathematical theory of stochastic partial differential equations, it can be applied to other physical problems that are also reduced to the problems concerning stochastic equations. It is of course a tedious thing to explain each of these sujects. Therefore we terminate this section only by noting that the theory is considered to have a close correspondence with the theory of non-equilibrium statistical mechanics, via the analogies, both physical and mathematical, (cf., U. Frisch [10]).

1. 2, Problems in the Dissertation.

We have explained in the previous section an outline of the theory of wave propagation in random media to find that problems in the theory are reduced to those of partial differential equations with random coefficients. We will consider in this dissertation the partial differential equations which have the "white noise" as their coefficients, and will study the initial value problems of these equations from the viewpoint of the theory of wave propagation in random media.

The equations that are treated in this dissertation are the followings ;

CASE I),
$$\frac{\partial}{\partial t} u(t,x;\omega) + a(t,x) \frac{\partial}{\partial x} u(t,x;\omega) = A(t,x)u(t,x;\omega) + b(t,x)W(t,\omega) .$$

CASE II),
$$\frac{\partial}{\partial t} u(t,x;\omega) + \{a(t,x) + W(t,\omega)\}\frac{\partial}{\partial x} u(t,x;\omega) = A(t,x)u(t,x;\omega) + B(t,x) .$$

where $W(t,\omega)$ is a stochastic process that is called the white noise.

Speaking formally, each of these equations may represent respectively a phenomenon of wave propagation in such a random medium that is disturbed by the white noise which enters into the transmission medium as an external noise, and a phenomenon of wave propagation in a random medium that fluctuates as a consequence of the instability of the medium itself, (due to a thermodynamical reason, for example). However in such interpretations there must be some comments on each case because they include the white noise which is an unreasonable quantity from the viewpoint of physics. Mathematically, the white noise can be understood as a derivative of the Brownian motion process in the sense of random distributions (cf., K. Ito [19]), which does not have an ordinary function as its sample path. Therefore, following this consideration, it may be possible to understand these formal equations also in the sense of random distribution. But we do not employ such a procedure because it may fail to conserve the physical correspondence between the interpretations for each equation stated above and the equations defined in such a procedure. We wish to understand these equations as mathematical idealizations of those which have a precise meaning as usual stochastic partial differential equations explained in the previous section and which can be considered to describe some physical phenomena. In other words, we wish each of these equations to understand as a limit equation of the followings respectively.

CASE I)' $$\frac{\partial}{\partial t} u^{\alpha}(t,x;\omega) + a(t,x) \frac{\partial}{\partial x} u^{\alpha}(t,x;\omega)$$

$$= A(t,x)u^{\alpha}(t,x;\omega) + b(t,x) \frac{d}{d\,t} Z^{\alpha}(t,\omega).$$

CASE II)' $$\frac{\partial}{\partial t} u^{\alpha}(t,x;\omega) + \{a(t,x) + \frac{d}{d t} Z^{\alpha}(t,\omega)\} \frac{\partial}{\partial x} u^{\alpha}(t,x;\omega) = A(t,x)u^{\alpha}(t,x;\omega) + B(t,x) \quad ,$$

where $\{Z^{\alpha}(t,\omega), t \geq 0\}_{\alpha>0}$ is a family of approximation processes to the Brownian motion process $Z(t,\omega)$, each of which has some nice properties from the viewpoint of physics, (a.s. smoothness, for example).

Thus in order to study the equations in CASE I and II keeping some realities in discussions, it is necessary to set a natural definition of solutions. Namely the first problem in this dissertation is

P. 1) : To set a natural definition under which the solution of each equation in CASE I and II can be understood as a limit, in a sense, of each corresponding solution of approximate equation in CASE I' and II'.

It will turn out to be clear in later discussions that the above problem is reduced to a problem of defining the new types of stochastic integrals which can be characterized as a limit of a sequence of the random Stieltjes integrals with respect to a family of approximation processes $\{Z^{\alpha}(t,\omega), t \geq 0\}_{\alpha=1}$.

After setting a natural definition of solutions, we shall be concerned with the next problem.

P. 2), The existence and the uniqueness properties of solutions of the initial value problems of these equations.

Following the general schem in the theory of wave propagation in random media, we shall investigate the statistical properties of solutions . At this stage, we will treat the different types of problems in each Case.

CASE I). The author studies the equation beeing stimulated by the recent progresses in the statistical theory of communications or in the theory of stochastic controls where it seems to be customary to postulate the " hypothesis of the additive noise " which describes that a signal transmitted is received as a sum of a certain noise. The author wishes to check the validity of this hypothesis and investigate the statistics of the additive noise, that is, the third problem in this case is

P. $3)_{(I)}$. To give a precise definition to the notion of the additive noise and to study the statistical properties of it in the linear system considered.

CASE II). As an application of the theory of wave propagation in random media to the theory of nonequilibrium statistical mechanics, it was pointed out by many authors that there is a close analogy between the random wave motion and the motion of Brownian paricle which is suspended in a heat bath, (cf., U. Frisch [10], Ford-Kac-Mazur [8]). The author expects the equation in this case to serve as a mathematical description to this analogy, or in other words, to stand for a transportation equation of a certain physical quantity carried by a diffusion particle. So it is desirable to check the following

P. $3)_{(II)}$. Does the mean value of a solution satisfy a parabolic equation ?

The dissertation is devoted to the discussions on these problems .

1. 3 Outline of the Dissertation .

In this dissertation we are concerned with the problems explained in the previous section. The dissertation consists of six chapters, conclusion and references. It can be devided into two parts. The first half (Chapter II, III) is concerned with the theory of B-derivatives and the new types of stochastic integrals of stochastic processes, which will provide us an answer to the problem, P. I). The latter half (Chapter IV, V) is concerned with problems in wave propagation in random media which correspond to the problems, P.2), $P.3)_{(I)}$, $P.3)_{(II)}$.

Herein we describe the outline of each chapter.

In Chapter II, we shall introduce the notion of B-derivatives of stochastic processes and discuss the properties of the derivatives and B-differentiable processes. We shall show a conjugate relation between the stochastic integration and the B-differentiation, and establish assertions concerning the integral representations of B-differentiable processes. Throughout the discussions in this dissertation, the results obtained in this chapter will serve as preliminaries.

In Chapter III, we shall introduce the new types of stochastic integrals $\{\mathcal{J}^{+}_{k}(f),\ 0\leq k\leq 1\}$, which include the Ito's integral as an example. Among them we restrict our attentions to $\mathcal{J}^{+}_{1/2}(f)$

and show the relation between the integral and the limit of a sequence of random Stieltjes integrals. We shall also compare the integral with other types of stochastic integrals such as the integral of Stratonovich and that of Stratonovich-Fisk. We will find in later discussions (Chapters IV,V) that the theory developed in Chapters II, III provides us an answer to the problem P.1).

In Chapter IV, we shall consider an initial value problem of the partial differential equation in CASE I, and establish the existence and the uniqueness theorems of solutions. On the basis of these results, we shall consider the problem $P.3)_{(I)}$ and introduce a precise definition of the additive noise of a linear system. After the discussion on the additive noise, we shall apply it to an exemple and find that there is a situation where we can estimate the values of variables of the transmission line by a practical observation of the additive noise.

In Chapter V, we shall consider an initial value problem of the partial differential equation in CASE II. We shall show the existence of solutions and verify that the averaged function of a solution constructed by the characteristic method becomes also a solution of an initial value problem of a linear parabolic (heat) equation, $(P.3)_{(II)})$.

In Chapter VI, we shall discuss the uniqueness question of stochastic solutions of the equation in CASE II. We will generalize the equation to a system of equations of the same type, and will establish a theorem concerning the uniqueness of averaged functions.

The conclusion of this dissertation will be noted in Chapter VII and references will be found at the end of this dissertation.

The context of Chapter II is mainly taken from the published papers [33], [29] of the Proc. Japan Acad., and partly from the published paper [31] of Z.Wahrscheinlichkeitstheorie, Chapter III from the published paper [30] of Proc. Japan Acad., Chapter IV from a paper in preparation [32], Chapter V from the published papers [31] and [33], Chapter VI from a paper [34] which is to appear in this year.

1.4 Comments on Notations and Terminologies.

In this dissertation, we shall extensively be concerned with discussions on stochastic processes, which will sometimes be called "random processes" or "random functions". When we consider a stochastic process $\{X_t(\omega),\ t\in I\}$ (I is a certain interval in R^1), it is always assumed to be R^1-valued and defined on a probability space $(\Omega, \mathfrak{F}, P)$. For a notational convenience, we shall often write them as $X_t(\omega)$, $X(t,\omega)$ or $X(t)$ when there may be no possibility of confusions. Moreover, throughout the discussions, we shall use the following terminologies :

(1), Expectations; The notation $E\{(X_t(\omega))^\alpha\}$ will stand for the expectation of the quantity $(X_t(\omega))^\alpha$ with respect to a prescribed probability measure P.

(2), Equalities; It is meant by the equalities between stochastic processes that relations are valid in the sense of the "stochastic equivalence". Namely the notation " $X(t,\omega) = Y(t,\omega)$ " is equivalent to the statement " $P\{X(t,\omega) = Y(t,\omega)\} = 1$, for each t. "

(3), Continuities; A stochastic process $\{X_t(\omega),\ t\in I\}$ will be called "stochastically continuous on I" provided that for an arbitrary positive number ε, there exists a positive number δ such that;

$P\{|X_t(\omega) - X_s(\omega)| > \varepsilon\} < \varepsilon$ for any t, s ($|t - s| < \delta$), in the interval I

A stochastic process $\{X_t(\omega), t \geq 0\}$ is also called " continuous in $L^n(\Omega)$-sense", or equivalently " $L^n(\Omega)$-continuous"when it posseses the property ; For an arbitrary positive number ε, there exists a positive number δ which assures the inequality $E\{|X_t(\omega) - X_s(\omega)|^n\} < \varepsilon$ for any t, s ($|t - s| < \delta$) in I.

(4), Separabilities; Stochastic processes are assumed to be separable in this dissertation, (cf., J. L. Doob [6]). Especially, when we write an equality between stochastic processes, we always understand it to hold between separable versions of each stochastic process, if necessary.

(5), Discontinuities; When we say that a stochastic process $\{X_t(\omega), t \in I\}$ has no discontinuities of the second kind on the interval I, we intend to mean by this that the stochastic process $X_t(\omega)$ has left, and right-hand limits $X_{t-}(\omega)$, $X_{t+}(\omega)$ with probability one at each t in I.

For the other special concepts about stochastic processes, such as "the martingales", "the Gaussian processes","the Brownian motion process", etc., we should refer to some texts on probability theory, (e.g., J. L. Doob [6], or K. Ito [21]).

CHAPTER II B-Derivatives of Stochastic Processes.

2. 1. Introduction.

As the classical calculus of integration and differentiations did in the theory of Newtonian mechanics, the theory of stochastic integrals and of stochastic differential equations may have a great deal to do with problems on stochastic dynamical systems, especially of those which have the Brownian motion as an element. In fact, we can find its application fields in such stochastic theories that are concerned with dynamical systems which possess the randomness in their features (e.g., the theory of stochastic controls, H. Kushner [26]), and in problems of statistical mechanics of particles that are suspended in a heat bath, (cf., S. Chandrasekhar [2], A. Einstein [7], Ornstein-Uhlenbeck [42]). However we can not be sure on the realities of such theories that are developed on the base of the Ito's theory of stochastic differential equations, while the theory of Brownian motion was originated from the purely physical considerations. The reasons which prevented the theory of Ito from the vivid applications to physical problems, may be that the theory itself is in so far a state to admit any physical interpretations, and that the theory is not convenient to work with for such practical purposes. For example, there are not any notion of differentiation which have a right to exist in itself in the theory of stochastic differential equations, though it is customary for us to describe a physical phenomenon using a notion of differentiation. (We should remind ourselves that the differential formula in the theory of Ito mean to say nothing more but symbolical abbreviations of stochastic integral equations.)

In this chapter we will consider a differentiation of stochastic processes with respect to the Brownian motion as a first step to make the Ito's theory to be applicable to the physical problems. As for a differentiation of stochastic processes, the discussions on it are very rare except those works of K. Ito [18], E. Nelson [28] and D. Isaacson [15]. In his monograph, " Dynamical Theory of Brownian Motion", E. Nelson introduced a notion of "mean derivative" of stochastic processes which corresponds to a usual notion of velocity in Newtonian mechanics. D. Isaacson considered in his paper [15], a derivative of a stochastic process with respect to a martingale. He considered the differentiation only of such a stochastic process that is represented by a stochastic integral with respect to a prescribed martingale, and verified that the differentiation of a stochastic integral yields a stochastic process that is the integrand of the integral. But he did not verify the complete duality between these operations. That is , he did not consider such a inverse problem; " Does the integration of the derivative yield an original stochastic process ?"

Hereafter we will introduce a notion of B-derivatives of stochastic processes, and will investigate in some detailes the properties of B-derivatives and B-differentiable processes in the next section, 2.2. In section 2.3, we shall consider about the duality between the stochastic integration and the B-differentiation. As a consequence of the consideration on the question of duality of these operations, we shall give two propositions which are concerned with the integral representations of B-differentiable processes.

The contents of this chapter are full expositions of those which were included in the author's ealier publications S. Ogawa [29], [31], and will serve as preliminaries for the discussions in this dissertation.

2. 2. B-Derivatives of Stochastic Processes

Let $\{Z(t,\omega),\ t\geq 0\}$ be an R^1-valued Brownian motion process defined on (Ω, F, P), a complete probability space, and let $\mathcal{N}_t^s$ be the smallest σ-algebra generated by $\{Z(\tau,\omega)-Z(s,\omega),\ s\leq\tau\leq t\}$.

<u>Definition 2. 1</u> Let $x_t(\omega)$, $t\geq 0$ be an R^1-valued, $\mathcal{B}_{[0,t]}\times\mathcal{N}_t^0$-measurable stochastic process. If for a positive integer n_0, there exists a stochastic process $\check{X}_t(\omega)$ $((t,\omega)\in[0,T]\times\Omega)$ satisfying the following conditions D.1) ∿ D.3), we shall call this "the B^+-derivative of $X_t(\omega)$ in the L_{2n}-sense" and denote the relation by

$$\check{X}_t(\omega) = \frac{\partial^+}{\partial^+ Z_t} X_t(\omega) .$$

D. 1) $\check{X}_t(\omega)$ is $\mathcal{B}_{[0,t]}\times\mathcal{N}_t^0$ - measurable

D. 2) $E\{|\check{X}_t(\omega)|^{2n}\}$ is bounded on $[0,T]$

D. 3) $\lim\limits_{s\downarrow t} E\left\{\frac{1}{\sqrt{s-t}}\{X_s(\omega) - X_t(\omega) - \check{X}_t(\omega)(Z_s(\omega) - Z_t(\omega))\}\right\}^{2n} = 0,$

for any t in $[0,T]$.

And if $X_t(\omega)$ has a B^+-derivative in L_{2n}-sense, on a certain interval $[a,b]$, we will say that $X_t(\omega)$ is $B^+(M_{2n})$-differentiable on the interval $[a,b]$.

For a $\mathcal{B}_{[t,T]} \times \mathcal{N}_T^{\tau}$ - measurable $(0<T<+\infty)$ stochastic process $Y_t(\omega)$ $(0 \leq t \leq T)$ its derivative with respect to the Brownian motion $Z(t,\omega)$ is defined in a similar way: If there exists a stochastic process $\check{Y}_t(\omega)$ $(0<\varepsilon \leq t \leq T)$ satisfying the following conditions $\tilde{D}.1) \sim \tilde{D}.3)$, we will call this the B^--derivative of $Y_t(\omega)$ in L_{2n}-sense on an interval $[\varepsilon,T]$ $(0<\varepsilon<T)$,

$\tilde{D}$. 1) $\check{Y}_t(\omega)$ is $\mathcal{B}_{[t,T]} \times \mathcal{N}_T^t$ - measurable for $t \in [\varepsilon,T]$.

$\tilde{D}$. 2) $E\{|\check{Y}_t(\omega)|^{2n}\}$ is bounded on $[\varepsilon,T]$.

$\tilde{D}$. 3) $\lim\limits_{s \uparrow t} E\{ \frac{1}{\sqrt{t-s}} \{Y_t(\omega) - Y_s(\omega) - \check{Y}_t(\omega)(Z_t(\omega) - Z_s(\omega))\}\}^{2n} = 0.$

And the relation between Y_t and $\check{Y}_t$ will be denoted in

$$\check{Y}_t(\omega) = \frac{\partial^-}{\partial^- Z_t} Y_t(\omega)$$

The notation $\frac{\partial}{\partial Z_t}$ is based on the following fact.

(Example 1). Let $x^{(s,x)}(t,\omega)$ $(t \geq s \geq 0)$ be the diffusion process determined by the following Itô's stochastic integral equation,

$$(2.1). \quad x_t^{(s,x)}(\omega) - x = \int_s^t a(\tau, x_\tau^{(s,x)}(\omega))d\tau + \int_s^t b(\tau, x_\tau^{(s,x)}(\omega))dZ_\tau(\omega),$$

$$(2.2) \quad x_s^{(s,x)}(\omega) = x \quad \text{a. s.,}$$

where a(t,x) and b(t,x) are such functions that are bounded and continuous in t, Lipschitz-continuous in x.

The solution $x_t^{(s,x)}(\omega)$ is $\mathcal{N}_t^s \times \mathcal{B}_{[s,t]}$-measurable in (t,ω) for any fixed s, and is $B^+(M_{2n})$-differentiable on any finite interval [s,u] with the following derivative, which is not difficult to verify ;

$$\frac{\partial^+}{\partial^+ Z_t} x_t^{(s,x)}(\omega) = b(t, x_t^{(s,x)}(\omega)).$$

Since the diffusion process $x_t^{(s,x)}(\omega)$ is $\mathcal{N}_t^s \times \mathcal{B}_{[s,t]}$ - measurable in (s,ω) for any fixed t, we can also consider its B^--derivatives. By a slight computation, we will see that it is $B^-(M_{2n})$-differentiable in s (n, a positive integer), and that the derivative is given as the solution of the following stochastic differential equation.

$$(2.3) \qquad X^t(s,\omega) + b(t,x) = \int_t^s a_x(u,X^{(t,x)}(u,\omega))\, X^t(u,\omega)du$$

$$+ \int_t^s b_x(u,X^{(t,x)}(u,\omega))\, X^t(u,\omega)dZ(u,\omega),$$

where $$X^t(s,\omega) = \frac{\partial^-}{\partial^- Z_s} X^{(t,x)}(s,\omega).$$

The next proposition assures that these derivatives are well defined.

<u>Proposition 2. 1</u> If a $\mathcal{B}_{[0,t]}\times\mathcal{N}^0_t$ ($\mathcal{B}_{[t,T]}\times\mathcal{N}^t_T$)-measurable stochastic process $X_t(\omega)$ is $B^+(B^-)(M_{2n})$-differentiable then its derivative $\frac{\partial^+}{\partial^+ Z_t} X_t(\omega)$ ($\frac{\partial^-}{\partial^- Z_t} X_t(\omega)$) is uniquely determined up to stochastic equivalence.

(Proof of Proposition 2. 1) Let $Y^1_t(\omega)$ and $Y^2_t(\omega)$ be B^+-derivatives of a stochastic process $X_t(\omega)$ in L_{2n}-sense. Then, from conditions D.1) and D.2) we have for any $s>t$ the next

$$E\{|Y^1_t(\omega) - Y^2_t(\omega)|^{2n}\} = \frac{1}{(2n-1)!!} E[|Y^1_t(\omega) - Y^2_t(\omega)|^{2n}$$

$$\cdot\times\ E\Big\{\Big(\frac{1}{\sqrt{s-t}}(Z_s(\omega) - Z_t(\omega))\Big)^{2n}\Big|\mathcal{N}^0_t\}].$$

Applying the Schwarz's inequality, we get

$$\leq KE\Big\{\frac{1}{\sqrt{s-t}}\{X_s - X_t - Y^1_t(Z_s - Z_t)\}\Big\}^{2n}$$

$$+ KE\Big\{\frac{1}{\sqrt{s-t}}\{X_s - X_t - Y^1_t(Z_s - Z_t)\}\Big\}^{2n}$$

, where K is a positive constant.

Taking s sufficiently close to t in the above estimate, we obtain $E\{(Y_t^1(\omega) - Y_t^2(\omega))^{2n}\} = 0$ for any fixed t by virtue of condition D.3). For the case of B^--derivatives, we will obtain the same conclusion in a similar way. And this completes the proof of Proposition.

As for the degrees of momenta in $B^{\pm}$-differentiations, we have the following

Proposition 2. 2 . (i) If a $\mathcal{B}_{[0,t]}\times\mathcal{N}_t^0(\mathcal{B}_{[t,T]}\times\mathcal{N}_T^t)$-measurable stochastic process $X_t(\omega)$ is $B^+(M_{2n})$ $(B^-(M_{2n}))$-differentiable for a certain positive integer n, at t, then the stochastic process $X_t(\omega)$ is also $B^+(M_{2m})$ $(B^-(M_{2m}))$-differentiable at t for m=1, 2, ..., n-1, and derivatives coincide with each other up to stochastic equivalence. (ii) If a stochastic process $X_t(\omega)$ is $B^+(M_2)$ $(B^-(M_2))$-differentiable at t, with the derivative $\check{X}_t(\omega)$ and if the following quantity

$$E\ \{\frac{1}{\sqrt{s-t}}X_s - X_t - X_t(Z_s - Z_t)\}\}^{2n}$$

$$(\text{ or } \quad E\ \{\frac{1}{\sqrt{t-s}}\{X_t - X_s - X_t(Z_t - Z_s)\}\}^{2n}\ )$$

remains finite for any $s\geq t$ $(t\geq s)$, then the same result as (i) holds for the process $X_t(\omega)$.

(Proof of Proposition 2. 2) It may suffice to verify the assertions (i), (ii), in the case of B^+-derivatives.

Let us put

$$(2.4) \qquad \delta(s,t) = \frac{1}{\sqrt{s-t}} \{X_s(\omega) - X_t(\omega) - \check{X}_t(\omega)\ (Z_s(\omega) - Z_t(\omega))\}$$

for a fixed t and for an arbitrary point $s(\geq t)$, where $\check{X}_t(\omega)$ is the B^+-derivative, mentioned in each assertion.

(i) By an application of the Hölder's inequality, we have

$$E\{(\delta(s,t))^{2(n-1)}\} \leq [E\{(\delta(s,t))^{2(n-2)}\}^{\frac{n}{n-2}}]^{\frac{n-2}{n}}$$

$$\times\ [E\{(\delta(s,t))^2\}^{\frac{n}{2}}]^{\frac{2}{n}}$$

$$\leq [E\{(\delta(s,t))^{2n}\}]^{\frac{n-2}{n}} \times [E\{|\delta(s,t)|^n\}]^{\frac{2}{n}}$$

$$\leq [E\{(\delta(s,t))^{2n}\}]^{\frac{n-2}{n}} \times [E\{(\delta(s,t))^{2n}\}]^{\frac{1}{n}}$$

here we have used the Schwarz's inequality at the last step. The estimate above shows that if $X_t(\omega)$ is $B^+(M_{2n})$- differentiable at t with the derivative $X_t(\omega)$, then it is also $B^+(M_{2(n-1)})$-differentiable with the same derivative. The assertion of (i) follows from a successive usage of the estimate and from Proposition 2.1.

(ii) Since we have

$$E\{|\delta(s,t)|^n\} \leq [E\{(\delta(s,t))^{2(n-1)}\}]^{\frac{1}{2}} \times [E\{(\delta(s,t))^2\}]^{\frac{1}{2}},$$

therefore from this estimate we get the following

$$[E\{(\delta(s,t))^{2(n-1)}\}]^{\frac{n-1}{n}} \leq E\{(\delta(s,t))^{2n}\}^{\frac{n-2}{n}} \times [E\{(\delta(s,t))^2\}]^{\frac{1}{n}} .$$

The assertion (ii) is a direct consequence of this inequality and of the result of Proposition 2.1. Q. E. D

As for the computational rules in the B-differentiation, we have the following assertion which is easy to verify.

Proposition 2. 3 (i) If the functions $f_t(\omega)$ and $g_t(\omega)$ are B^+(or B^-)(M_{2n})-differentiable, then the linear combination $C_1f_t(\omega) + C_2g_t(\omega)$ (C_1,C_2; constants) is also $B^+(B^-)(M_{2n})$-differentiable and their derivatives satisfy

$$(2.5), \qquad \frac{\partial^{\pm}}{\partial^{\pm} Z_t}\{C_1 f_t(\omega) + C_2 g_t(\omega)\} = C_1\frac{\partial^{\pm}}{\partial^{\pm} Z_t} f_t(\omega) + C_2\frac{\partial^{\pm}}{\partial^{\pm} Z_t} g_t(\omega) .$$

(ii) If the functions $f_t(\omega)$, $g_t(\omega)$ are $B^+(B^-)(M_{4n})$-differentiable, then the product $(f_t \cdot g_t)(\omega)$ is $B^+(B^-)(M_{2n})$-differentiable and the derivatives satisfy the following

$$(2.6). \qquad \frac{\partial^{\pm}}{\partial^{\pm} Z_t}(f_t \cdot g_t)(\omega) = f_t(\omega)\frac{\partial^{\pm}}{\partial^{\pm} Z_t} g_t(\omega) + g_t(\omega)\frac{\partial^{\pm}}{\partial^{\pm} Z_t} f_t(\omega) .$$

Now let us investigate the properties of B-differentiable processes. The first thing that we can say is the following

Propositio 2. 4. If a stochastic process $\{X_t(\omega),\ t\geq 0\}$ is $B^+(B^-)$ (M_{2n})-differentiable ($n\geq 1$) on the interval $[0,T]$, then the process is stochastically right (left)-continuous at any t in $[0,T]$.

(Proof) We will give a proof only for the B^+-differentiable case.

From condition D.3), we know that for a fixed $t_0 (\in [0,T])$, and an arbitrary positive number ε, there exists a positive number δ such that,

$$E\{(X_t - X_{t_0} - \dot{X}_{t_0}(Z_t - Z_{t_0}))^{2n}\} \leq \varepsilon (t - t_0)^n$$

for any t in $[t_0, t_0 + \delta]$, where $\dot{X}_t(\omega) = \frac{\partial^+}{\partial^+ Z_t} X_t(\omega)$.

Therefore

$$(2.7) \qquad E\{(X_t - X_{t_0})^{2n}\} \leq (\varepsilon + A)(t - t_0)^n,$$

where $A = (2n-1)!!\ E\{|\dot{X}_{t_0}|^{2n}\}$ which is finite by virtue of condition D.2).

On the other hand, from the Tchebychev's inequality, we have

$$P\{|X_t - X_{t_0}| > \varepsilon\} < \frac{1}{\varepsilon^{2n}}(\varepsilon + A)(t - t_0)^n, \quad \text{for} \quad \delta > t - t_0 > 0,$$

which states that X_t is stochastically right-continuous at $t = t_0$.

Q. E. D.

The estimate (2.7) seems to say that a stochastic process which is $B^+(B^-)(M_{2n})$-differentiable for $n \geq 2$, is almost surely right-continuous(or left-continuous). However, it is hard to prove that, because the constants in the estimate depend on the point t_0.

In order to yield a almost sure continuity of a B-differentiable stochastic process, it may be necessary to assume that the expression (2.7) holds uniformly in t_0. That is, we are reached to the following.

<u>Definition 2. 2</u> A $\mathcal{B}_{[0,t]}\times \mathcal{N}_t^0(\mathcal{B}_{[t,T]}\times\mathcal{N}_T^t)$-measurable stochastic process $X_t(\omega)$ is called uniformly $B^+(B^-)(M_{2n})$-differentiable on the interval $[0, T]$ if it satisfies D.1), D.2) (or $\tilde{D}$.1), $\tilde{D}$.2)) and the followings respectively

$$D^+_. 3)' \qquad \lim_{h\downarrow 0} \sup_{\substack{0<s-t\leq h \\ 0\leq t\leq T}} E\{\frac{1}{\sqrt{s-t}}\{X_s(\omega) - X_t(\omega) - \check{X}_t(\omega)(Z_s(\omega) - Z_t(\omega))\}\}^{2n} =$$

$$D^-_. 3)' \qquad \lim_{h\downarrow 0} \sup_{\substack{0<t-s\leq h \\ 0\leq t\leq T}} E\{\frac{1}{\sqrt{t-s}}\{X_t(\omega) - X_s(\omega) - \check{X}_t(\omega)(Z_t(\omega) - Z_s(\omega))\}\}^{2n}.$$

It is obvious that a uniformly $B^+(B^-)(M_{2n})$-differentiable process is also $B^+(B^-)(M_{2n})$-differentiable with the same derivatives.

<u>Proposition 2. 5</u> A separable and uniformly $B^+(B^-)(M_{2n})$-differentiable stochastic process ($n\geq 2$) is almost surely continuous.

(Proof) Let $X_t(\omega)$ be uniformly $B^+(M_{2n})$-differentiable on the interval $[0,T]$, with $\check{X}_t(\omega)$ its derivative. Then, from condition $D^+_.3)'$, we know that there exists a positive number δ

for any positive number ε, such that

$$E\{(X_s - X_t - \check{X}_t(Z_s - Z_t))^{2n}\} \leq \varepsilon(s-t)^n$$

holds for any $s>t$, $(s-t<\delta,\ 0\leq t\leq T)$.

Therefore we have

$$E\{(X_s - X_t)^{2n}\} \leq (\varepsilon+A)(s-t)^n \tag{2.8}$$

for any $s>t$ $(s-t<\delta)$

Since the uniformly $B^+(M_{2n})$-differentiable process is stochastically continuous by virtue of Proposition 2.4, and since a measurable, stochastically continuous process has no discontinuities of the second kind, the stochastic process $X_t(\omega)$ has both left-hand and right-hand limits at any point in $[0,T]$.

Let $N(\varepsilon)$ be the number of points t at which $|X_{t+0} - X_{t-0}| > 2\varepsilon$ holds. Let us consider a family of partitions $\{\Delta^{(n)}\} = \{0\leq t_1^{(n)} < t_2^{(n)} < \dots < t_n^{(n)} \leq T\}$ of the interval $[0,T]$ which possesses the property $\lim_{n\to\infty} \max_{1\leq i\leq n} (t_{i+1}^{(n)} - t_i^{(n)}) = 0$. and let $N^{(n)}(\varepsilon)$ be such that

$$N^{(n)}(\varepsilon) = \#\{i;\ |X(t_{i+1}^{(n)}) - X(t_i^{(n)})| > \varepsilon\}$$

Then it is clear that $\underline{\lim}_{n\to\infty} N^{(n)}(\varepsilon) \geqslant N(\varepsilon)$, and that

$$EN^{(n)}(\varepsilon) = \sum_{i=1}^{n} P\{|X(t_{i+1}^{(n)}) - X(t_i^{(n)})| > \varepsilon\}$$

On the other hand, for a sufficiently large n that yields $\max_{1\leq i\leq n} (t_{i+1}^{(n)} - t_i^{(n)}) < \delta$, we have from (2.8) the next

$$(2.9)\qquad P\{|X(t_{i+1}^{(n)}) - X(t_i^{(n)})| > \varepsilon\} \leq \frac{(\varepsilon+A)}{\varepsilon^{2n}} (t_{i+1}^{(n)} - t_i^{(n)})^n .$$

Therefore, we have

$$\lim_{n\to\infty} EN^{(n)}(\varepsilon) \leqq \lim_{n\to\infty} \sum_{n=p}^{n} \frac{(\varepsilon+A)}{\varepsilon^{2n}} (t_{i+1}^{(n)} - t_i^{(n)})^n = 0$$

by virtue of condition D.2).

Hence, we know by the Fatou's lemma,

$$EN(\varepsilon) \leqq E \underline{\lim}_{n\to\infty} N^{(n)}(\varepsilon) \leq \lim_{n\to\infty} EN^{(n)}(\varepsilon) = 0.$$

That is, $N(\varepsilon) = 0$ with probability one for an arbitrary positive number ε. This completes the proof.

The necessary and sufficient condition for a $B^+(B^-)(M_{2n})$-differentiable process to be a uniformly $B^+(B^-)(M_{2n})$-differentiable process is unknown. But we can show a sufficient condition in the next proposition whose result will perform a helpful role throughout the discussions in this dissertation.

<u>Proposition 2. 6</u> If an almost surely continuous stochastic process $X_t(\omega)$ is $B^+(M_{2n})$-differentiable $(n\geq 1)$ on the interval

$[0,T]$, and if its $B^+(M_{2n})$-derivative is almost surely continuous on $[0,T]$, then $X_t(\omega)$ is uniformly $B^+(M_{2n})$-differentiable on $[0,T]$.

(Proof) Let us put $\mathcal{E}(t,s) = E[(\delta(s,t))^{2n}]$, where $\delta(s,t)$ is a quantity defined in(2.5).

Then the continuities of $X_t(\omega)$ and $\check{X}_t(\omega)$ yield that the function $\mathcal{E}(t,s)$ is continuous in t $(0\leq t<s)$ for a fixed s and is continuous in s $(t\leq s\leq T)$ for a fixed t.

Therefore the next function $\bar{\mathcal{E}}(t,s)$,

$$\bar{\mathcal{E}}(t,s) = \begin{cases} \mathcal{E}(t,s) & \text{for } 0\leq t<s\leq T \\ 0 & \text{otherwise} \end{cases}$$

is continuous in (t,s) on the rectangle, since we have

$\lim_{s\downarrow t} \mathcal{E}(t,s) = 0$ for any fixed t. In fact, if it is not a continuous function, then its discontinuity must occur in such a manner; $\exists\, t_0$, such that $|\bar{\mathcal{E}}(t_0-0,t_0) - \bar{\mathcal{E}}(t_0,t_0)| > \varepsilon\ (>0)$

that is, we must have $\lim_{t \uparrow t_0} \varepsilon(t,t_0) > a$ for some t_0 and $a(>0)$.

On the other hand, for an arbtrary positive number $\varepsilon_1 (< a)$, there exists a point $t_1 < t_0$ such that $|\varepsilon(t_1,t_0) - \varepsilon(t_0-0,t_0)| < \varepsilon_1$ by virtue of the continuity of $\varepsilon(t,s)$ in t. That is, we have $\varepsilon_1 + \varepsilon(t_0-0,t_0) > \varepsilon(t_1,t_0) > \varepsilon(t_0-0,t_0) - \varepsilon_1 > a - \varepsilon_1$. Since a is a positive constant and since we have $\varepsilon(t_1,t_1) = 0$ this contradicts the assumption of continuity of $\varepsilon(t,s)$ in s. Therefore $\bar{\varepsilon}(t,s)$ is a uniformly continuous function on $[0,T] \times [0,T]$.

Hence, for any positive number ε, there exists a positive number δ such that

$$(2.10) \qquad |\bar{\varepsilon}(t_1,s_1) - \bar{\varepsilon}(t_2,s_2)| < \varepsilon \qquad \text{for any pairs } (t_1,s_1),$$

(t_2,s_2) such that $|t_1 - t_2| < \delta$, $|s_1 - s_2| < \delta$.

Thus taking $s_1 = t_2 = s_2$ we obtain from (2.10) the following relation;

$$\varepsilon(t_2,t_1) = \bar{\varepsilon}(t_2,t_1) < \varepsilon \qquad \text{for any } 0 < t_1 - t_2 < \delta,$$

and this completes the proof.

(Remark) The assumption of almost sure continuity of $\check{X}_t(\omega)$ may be too strong. In fact we have the following exemple.

<u>Exemple</u> Let $f_t(\omega)$ be a random function such that

(i) $f_t(\omega)$ is $\mathcal{B}_{[0,t]} \times \mathcal{N}_t^0$- measurable,

(ii) $\displaystyle\int_0^T E[(f_t(\omega))^2]\, dt < \infty$.

Then it is clear that the following stochastic process

(2.11), $$F_t(\omega) = \int_0^t f_s(\omega)\, dZ(s,\omega),$$

is $B^+(M_2)$-differentiable for almost every t in $[0,T]$.

Moreover, we can see that if the function $f_t(\omega)$ is $L^2(\Omega)$-continuous on $[0,T]$, then the stochastic process $F_t(\omega)$ is uniformly $B^+(M_2)$-differentiable on $[0,T]$.

2. 3. Integral Representations of B^+-differentiable Processes.

In the previous section, we have investigated some properties of B-derivatives and of B-differentiable stochastic processes. Especially in the final example, we have checked that the B^+-differentiation performs like a dual operation of the stochastic integration. Here we are concerned with a further discussion on this duality, that is,we are interested in the following inverse problem; "Let $X_t(\omega)$ be a uniformly $B^+(M_{2n})$-differentiable stochastic process with $\check{X}_t(\omega)$ its B^+-derivative. Then does the stochastic integration of $\check{X}_t(\omega)$ with respect to the Brownian motion process $Z(t,\omega)$, yield the original process $X_t(\omega)$?".

It is easy to propose counter examples to this conjecture, however we have the following case.

<u>Proposition 2. 7</u> If a square integrable mantingale $\{X_t, \mathcal{N}_t^0\}$ is uniformly $B^+(M_{2n})$-differentiable on a interval $[a,b]$ for some $n(\geq 1)$, then the stochastic process has a integral representation as follows;

$$(2.12) \qquad X_t(\omega) - X_a(\omega) = \int_a^t \check{X}_s(\omega)\, dZ(s,\omega), \quad \text{a.s. for each } t \text{ in } [a,b].$$

(Proof) It suffices to check that the relation

$$\delta(t) = E\,\{(X_t(\omega) - X_a(\omega) - \int_a^t \check{X}_s(\omega)\, dZ(s,\omega))^2\} = 0$$

holds for each t.

Let $\{\Delta^{(n)};\ a=t_1^{(n)} < t_2^{(n)} < \ldots < t_n^{(n)} = t\}_{n=1}^{\infty}$ be a family of partitions of the interval $[a,t]$ that satisfies the condition, $\lim_{n\to\infty} \max_{1\le i\le n} (t_{i+1}^{(n)} - t_i^{(n)}) = 0$. Now we have,

$$(2.13)\qquad \delta(t) = E\{(\sum_{i=1}^{n} (X_{t_{i+1}^{(n)}} - X_{t_i^{(n)}}) - \int_a^t \check{X}_s(\omega)\, dZ(s,\omega))^2\}$$

$$\leq 2E\{(\sum_{i=1}^{n} (X(t_{i+1}^{(n)}) - X(t_i^{(n)}) - \check{X}(t_i^{(n)})\, \Delta_i^{(n)} Z))^2\}$$

$$+ 2E\{(\int_a^t \check{X}_s(\omega)\, dZ(s,\omega) - \sum_{i=1}^{n} \check{X}(t_i^{(n)})\, \Delta_i^{(n)} Z)^2\},$$

where $\Delta_i^{(n)} Z = Z(t_{i+1}^{(n)},\omega) - Z(t_i^{(n)},\omega)$.

It is obvious from the definition of stochastic integrals (K. Ito [21]), that there exists a positive number n_1 for an arbitrary positive number ε, which yields

$$(2.14)\qquad E\{(\int_a^t \check{X}_s(\omega)\, dZ(s,\omega) - \sum_{i=1}^{n} \check{X}(t_i^{(n)})\, \Delta_i^{(n)} Z)^2\} < \varepsilon$$

for all $n \ge n_1$.

As for the remaining term in the above estimate, we have

$$E\{(\sum_{i=1}^{n}(X(t_{i+1}^{(n)}) - X(t_i^{(n)}) - \check{X}(t_i^{(n)})\,\Delta_i^{(n)}\,Z))^2\}$$

$$= \sum_{i=1}^{n} E\{(X(t_{i+1}^{(n)}) - X(t_i^{(n)}) - \check{X}(t_i^{(n)})\,\Delta_i^{(n)}\,Z)^2\}$$

$$+ 2\sum_{i>j}^{n} E\{(X(t_{i+1}^{(n)}) - X(t_i^{(n)}) - \check{X}(t_i^{(n)})\,\Delta_i^{(n)}\,Z)$$

$$\times\ (X(t_{j+1}^{(n)}) - X(t_j^{(n)}) - \check{X}(t_j^{(n)})\,\Delta_j^{(n)}\,Z)\}$$

$$= \sum_{i=1}^{n} E\{(X(t_{i+1}^{(n)}) - X(t_i^{(n)}) - \check{X}(t_i^{(n)})\,\Delta_i^{(n)}\,Z)^2\}$$

$$+ 2\sum_{i>j}^{n} E[\{X(t_{j+1}^{(n)}) - X(t_j^{(n)}) - \check{X}(t_j^{(n)})\,\Delta_j^{(n)}\,Z\}$$

$$\times\ E\{(X(t_{i+1}^{(n)}) - X(t_i^{(n)}) - \check{X}(t_i^{(n)})\,\Delta_i^{(n)}\,Z)|\mathcal{N}^0_{t_j^{(n)}}\}]$$

$$= \sum_{i=1}^{n} E\{(X(t_{i+1}^{(n)}) - X(t_i^{(n)}) - \check{X}(t_i^{(n)})\,\Delta_i^{(n)}\,Z)^2\} \quad ,$$

since we have

$$E\{X(t_{i+1}^{(n)}) - X(t_i^{(n)})|\mathcal{N}^0_{t_j^{(n)}}\} = 0,$$

$$E\{\check{X}(t_i^{(n)})\,\Delta_i^{(n)}\,Z\,|\mathcal{N}^0_{t_j^{(n)}}\} = E\{\check{X}(t_i^{(n)})\,E(\Delta_i^{(n)}\,Z|\mathcal{N}^0_{t_i^{(n)}})|\mathcal{N}^0_{t_j^{(n)}}\}\}$$

$$= 0 \ ,$$

by virtue the assumption, imposed on $\{X^t, \mathcal{N}^0_t\}$.

On the other hand, the uniform $B^+(M_{2n})$-differentiability of $X_t(\omega)$ assures the existence of a positive number n_2, such that

$$(2.15) \qquad E\,\{(X(t_{i+1}^{(n)}) - X(t_i^{(n)}) - \check{X}(t_i^{(n)})\,\Delta_i^{(n)}\,Z\,)^2\} < \varepsilon(t_{i+1}^{(n)} - t_i^{(n)}),$$

for an arbitrary $\varepsilon>0$, and for all $n\geq n_2$.

Hence, from (2.13), (2.14) and (2.15), we obtain the next taking sufficiently large $n>n_1, n_2$

$$(2.16) \qquad \delta(t) \leq 2\varepsilon\,(1 + (t-a)).$$

The arbitrariness of ε in (2.16) yields the result.

Q. E. D.

(Remark) The formula (2.12) shows that the assertion of (Proposition 2.5), which is concerned with the continuity of B^+-differentiable processes, still remains valid for uniformly $B^+(M_2)$-differentiable processes, provided that they are square integrable mantingales, with respect to a family $\{\mathcal{N}_t^0\}_{t\geq 0}$.

In order to proceed a consideration to more general case, it seems to be neccessary to introduce the notion of "mean derivatives of stochastic processes" which was proposed by E. Nelson in his manuscript [28]. Following E. Nelson, we introduce the

Definition An R^1-valued, $B_{[0,t]} \times \mathcal{N}_t^0$ - measurable stochastic process $\{X_t(\omega),\ t\in[0,T]\}$ is called M-differentiable, if it satisfies the followings:

(M.1) Each $X_t(\omega)$ is in $L^1(\Omega)$ and $t\to X_t(\omega)$ is continuous from $[0,T]$ into $L^1(\Omega)$.

(M.2) For each t in $[0,T]$,

$$DX(t,\omega) = \lim_{h\to 0+} E\{\frac{X(t+h,\omega)-X(t,\omega)}{h} | \mathcal{N}_t^0 \}$$

exists as a limit in $L^1(\Omega)$ and $t\to DX(t,\omega)$ is continuous from $[0,T]$ into $L^1(\Omega)$. And the random variable $DX(t,\omega)$ is called the "mean forward derivative of $X(t,\omega)$".

Heareafter we will shortly call $DX(t,\omega)$ "the M-derivative of X_t". Since $DX(t,\omega)$ is defined as above, it is of course $\mathcal{N}_t^0$ - measurable for each t.

Proposition 2. 8 If a stochastic process $\{X_t(\omega),\ t\geq 0\}$ is uniformly $B^+(M_{2n})$-differentiable $(n\geq 1)$ and M-differentiable on a interval $[a,b]$, with $X(t,\omega)$, $DX(s,\omega)$ its B^+-, M-derivatives respectively, and if $DX(t,\omega)$, is Riemann integrable in $L^2(\Omega)$-sense , then it has the integral representation of the following type.

$$(2.17)\qquad X(t,\omega) - X(a,\omega) = \int_a^t DX(s,\omega)ds + \int_a^t \check{X}(s,\omega)dZ(s,\omega)$$

, almost surely for each t in $[a,b]$.

(Remark) Here it must be noticed that the integral of $DX(t,\omega)$ exists as a Riemann integral in $L^1(\Omega)$-sense, since $t\to DX(t,\omega)$ is continuous in $L^1(\Omega)$.

(Proof) Let us put

$$\delta(t) = E\left\{\left(X(t,\omega) - X(a,\omega) - \int_a^t DX(s,\omega)ds - \int_a^t \check{X}(s,\omega)dZ(s,\omega)\right)^2\right\}$$

for a fixed t in $[a,b]$.

And let $\{\Delta^{(n)}\}_{n=1}^{\infty}$ be a family which was used above.

Then we have,

$$(2.18)\qquad \delta(t) \le 2E\left\{\left\{\sum_{i=1}^{n} \left(X(t_{i+1}^{(n)}) - X(t_i^{(n)}) - \check{X}(t_i^{(n)})\,\Delta_i^{(n)}Z - \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} DX(s,\omega)dZ\right\}^2\right\} + 2E\left\{\left(\int_a^t \check{X}(s,\omega)dZ(s,\omega) - \sum_{i=1}^{n} \check{X}(t_i^{(n)},\omega)\,\Delta_i^{(n)}Z\right)^2\right\}.$$

From a similar consideration to the previous case, it will turn out that $\delta(t) = 0$. In fact, as for the first term on the right hand side in (2,18), we have,

$$E\{\{\sum_{i=1}^{n}(X(t_{i+1}^{(n)}) - X(t_i^{(n)}) - \check{X}(t_i^{(n)})\Delta_i^{(n)} Z - \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} DX(s,\omega)ds)\}^2\}$$

$$= \sum_{i=1}^{n} E\left\{(X(t_{i+1}^{(n)}) - X(t_i^{(n)}) - \check{X}(t_i^{(n)})\Delta_i^{(n)} Z - \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} DX(s,\omega)ds)^2\right\}$$

$$+ 2\sum_{i>j}^{n} E\{(X(t_{j+1}^{(n)}) - X(t_j^{(n)}) - \check{X}(t_j^{(n)})\Delta_j^{(n)} Z - \int_{t_j^{(n)}}^{t_{j+1}^{(n)}} DX(s,\omega)ds)$$

$$\times E\left\{(X(t_{i+1}^{(n)}) - X(t_i^{(n)}) - \check{X}(t_i^{(n)})\Delta_i^{(n)} Z - \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} DX(s,\omega)ds \,\middle|\, \mathcal{N}^0_{t_j^{(n)}}\right\}\}.$$

$$= \sum_{i=1}^{n} E\,\{(X(t_{i+1}^{(n)}) - X(t_i^{(n)}) - \check{X}(t_i^{(n)})\,\Delta_i^{(n)} Z - \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} DX(s,\omega)ds\,)^2\}\,,$$

since we know by virtue of the lemma stated later, that

$$E\,\{(X(t_{i+1}^{(n)}) - X(t_i^{(n)}) - \check{X}(t_i^{(n)})\,\Delta_i^{(n)} Z - \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} DX(s,\omega)ds\,)|\mathcal{N}^0_{t_j^{(n)}}\}$$

$$= E\,\{E\,\{(X(t_{i+1}^{(n)}) - X(t_i^{(n)}) - \check{X}(t_i^{(n)})\,\Delta_i^{(n)} Z - \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} DX(s,\omega)ds\,)|\mathcal{N}^0_{t_i^{(n)}}\}|\mathcal{N}^0_{t_j^{(n)}}\}$$

$$= E\,\{E\,\{X(t_{i+1}^{(n)}) - X(t_i^{(n)}) - \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} DX(s,\omega)ds\,|\mathcal{N}^0_{t_i^{(n)}}\}\mathcal{N}^0_{t_j^{(n)}}\} = 0$$

On the other hand, we have

$$(2.19) \quad \sum_{i=1}^{n} E\{(X(t_{i+1}^{(n)}) - X(t_i^{(n)}) - \check{X}(t_i^{(n)})\,\Delta_i^{(n)} Z - \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} DX(s,\omega)ds\,)^2\}$$

$$\leq 2 \sum_{i=1}^{n} E\{(X(t_{i+1}^{(n)}) - X(t_i^{(n)}) - \check{X}(t_i^{(n)})\,\Delta_i^{(n)} Z\,)^2\}$$

$$+ 2 \sum_{i=1}^{n} E\{(\int_{t_i^{(n)}}^{t_{i+1}^{(n)}} DX(s,\omega)ds\,)^2\} .$$

Taking *n sufficiently large*, we can make the terms of the right hand side arbitrarily small by virtue of the uniform $B^+(M_{2n})$-differentiability of $X(t,\omega)$ and the Riemann-integrability of $DX(t,\omega)$ in $L^2(\Omega)$-sense. By this reason, together with estimates (2.18), (2.19) and the following lemma, we complete the proof.

(Lemma) (E. Nelson) Let $\{X_t(\omega),\ t\geq 0\}$ be an M-differentiable stochastic process on an interval $[a,b]$. Then we have

$$E\{X(t,\omega) - X(a,\omega)\,|\,\mathcal{N}_a^0\} = E\{\int_a^t DX(s,\omega)ds\,|\,\mathcal{N}_a^0\}.$$

For the proof of Lemma, we can find it in E. Nelson [28].

We can find a similar result in E. Nelson's monograph ([28]). The result that he obtained is as follows.

<u>Proposition</u> (E. Nelson) Let a M-differentiable stochastic process $\{X_t(\omega),\ t\geq 0\}$ satisfy

$$(2.20) \quad \delta^2(t) = \lim_{\Delta t\to 0+} E\,\{\frac{[X(t+\ t)-X(t)]^2}{\Delta t} \mid \mathcal{N}_t^0\}$$

exists in $L^1(\Omega)$ and is $L^1(\Omega)$-contimuous in t and such that $\delta^2(t) > 0$ for almost every t in $[0,T]$,

Then, the next relation holds for any t in $[0,T]$.

$$(2.21) \qquad X(t,\omega) - X(0,\omega) = \int_0^t DX(s,\omega)ds + \int_0^t \delta(s)dZ(s,\omega).$$

Comparing this with our result, we get the following assertion which states a sufficient condition for a M-differentiable process to be uniformly $B^+(M_{2n})$-differentiable on $[0,T]$.

<u>Proposition 2. 9</u> Let $\{X_t(\omega),\ t\geq 0\}$ be M-differentiable process,(differentiable on $[0,T]$). Then the stochastic process $\{X_t(\omega),\ t\geq 0\}$ is uniformly $B^+(M_2)$-differentiable, if it satisfies the Nelson's condition (2.20).

In other words the proposition above tells us, that our result in Proposition 2. 8 is more general than that of E. Nelson.

2. 4. Concluding Remarks

We are restricted our attentions to R^1-valued stochastic processes. The extensions of discussions to a multi-dimensional case may be carried directly. Let us try to set up the things for a multi-dimensional case.

Let $X(t,\omega) = (X^1(t,\omega), X^2(t,\omega), \ldots, X^n(t,\omega))$ be an R^n-valued stochastic process, and let $Z(t,\omega) = (Z^1(t,\omega), Z^2(t,\omega), \ldots, Z^m(t,\omega))$ be the R^m-valued Brownian motion processes, components of which are independent of each other. We construct a σ-field $\mathcal{N}_t^0$, by $\mathcal{N}_t^0 = \bigvee_{i=1}^{m} \mathcal{N}_t^{(i)}$, where $\mathcal{N}_t^{(i)}$ is the smallest σ-field generated by the sets $\{Z_s^i(\omega), s\leq t\}$. Then the direct analogy to the one-dimensional case, will lead us to the next

<u>Definition</u>. For a $B_{[0,t]}\times\mathcal{N}_t^0$ - measurable stochastic process $\{X_t(\omega), t>0\}$. if there exists a random martix $\check{X}_t(\omega) = \{\check{X}_t^{ij}(\omega), t\geq 0\}_{i,j}$, each components of which satisfies following conditions.

(M.1) $\check{X}_t^{i,j}(\omega)$ $(i=1,2,\ldots,m,\ j=1,2,\ldots,n)$ is $\mathcal{B}_{[0,t]}\times\mathcal{N}_t^0$-measurable

(M.2) $E\{|\check{X}_t^{i,j}(\omega)|^2\}$ is bounded on $[0,T]$.

(M.3) For each t in $[0,T]$, the next holds.

$$\lim_{s \downarrow t} E \{ \| \frac{1}{\sqrt{s-t}} (X_s - X_t - (\check{X}_t)(Z_s - Z_t)) \|^2_{R^n} \} = 0 ,$$

where $\{(X_t)(Z_s - Z_t)\}_i = \sum_{j=1}^{m} \check{X}_t^{ij} (Z_s^j - Z_t^j)$ and

$$\| Z \|^2_{R^n} = \sum_{i=1}^{n} |Z^i|^2 \quad \text{for any } R^n\text{-valued vector } Z.$$

or equivalently,

$$\text{(M.3)}' \qquad \lim_{s \downarrow t} E \{ \{ \frac{1}{\sqrt{s-t}} (X_s^i - X_t^i - \sum_{j=1}^{m} \check{X}_t^{ij} (Z_s^j - Z_t^j)) \}^2 \} = 0$$

for $i=1, 2, \ldots, n$.

Then the matrix $(\check{X}_t) = \{\check{X}_t^{ij}\}$ is called the $B^+(M_2)$-derivative of $X_t(\omega)$, and the process $\{X_t(\omega), t \geq 0\}$ is called B^+-differentiable on $[0,T]$.

To see that the definition is a natural one, we consider the following.

(Example) Let $X_t(\omega)$ be the solution of the following stochastic integral equation.

$$\text{(2.22)} \qquad X_t(\omega) - X_0 = \int_0^t A(s, X_s(\omega))ds + \int_0^t B(s, X_s(\omega))dZ_s^{tr}(\omega) \quad ,$$

where X_0 is an R^n-vector, $A(s, X_s(\omega)) = (A^1(s, X_s(\omega)), \ldots,$

$A^n(s, X_s(\omega)))$ and $B(s, X_s(\omega))$ is an $n\times m$ matrix $\{B^{ij}(s, X_s(\omega))\}_{\substack{1\leq i\leq n\\ 1\leq j\leq m}}$ and $Z_t^{tr}(\omega)$ is the transposed vector of $Z_t(\omega)$.

Under the sufficient conditions on regularities of A and B, we can see that $X_t(\omega)$ is $B^+(M_2)$-differentiable with the following derivative matrix $\check{X}_t(\omega)$,

$$(2.23) \qquad \check{X}_t(\omega) = \{B^{ij}(t, X_t(\omega))\}_{\substack{1\leq i\leq n\\ 1\leq j\leq m}} .$$

In this case, we may be able to establish various assertions as in one-dimensional case. However it is not yet known whether there are special phenomena which are inherent in such a multi-dimensional case.

We may also be able to extend the discussions for cases of square integrable martingales. The author expect that analogous considerations on such cases, may produce various fruitful results such as, the integral representations for a certain additive process or a Markov process. But the investigations on them are still in a "terra incognita".

CHAPTER III. The New Types of Stochastic Integrals.

3. 1. Introduction.

In his famous articles [16], [17], Professor K. Ito introduced the theory of stochastic integrals and originated the theory of stochastic differential equations which are now well known as the Ito's theory of stochastic integrals and of stochastic equations. Starting from his works, the theory of stochastic differential equations has been developed and used in various directions, not only by pure mathematicians but also by apllied mathematicians who are interested in the analyses of random phenomena. Especially in some fields of technology such as stochastic controls, or statistical theory of communications, the theory of Ito has been extensively apllied to investigate the problems about stochastic dynamical systems whose behaviors are characterized by the following Langevin types stochastic equations of ;

$$(* 1), \quad \begin{cases} \dfrac{d}{dt} X(t) = f(t, X(t)) + g(t, X(t))W(t,\omega), \\ X(0) = X_0 , \end{cases}$$

where $W(t,\omega)$ is a random process, called the "white noise" by engineers.

In such fields of applied mathematics, it has been customary to start discussions on the dynamical system , regarding that the formal expression (* 1) is the symbolical notation of the following stochastic integral equation ;

$$(* \ 2) \qquad X(t,\omega) - X_0 = \int_0^t f(s, X(s,\omega))ds + \int_0^t g(s, X(s,\omega))dZ(s,\omega) ,$$

where $\{Z(t,\omega) , t \geq 0\}$ is a Brownian motion process, and the second term on the right hand side is the Ito's stochastic integral.

Once the mathematical interpretation to the formal expression (* 1) had been given as above, it was a direct application of the Ito's theory to study the problems in random systems. Though the theory of Ito was not introduced for the aim of representing the physical phenomena, no one had been careful of the physical fidelity of such mathematical treatment until 1963. Questions on the physical fidelity were first considered by the russian mathematician R. L. Stratonovich in 1963 ([37]). His answer to the question was that a stochastic differential eqation (* 1), which was understood via the interpretation (*2), could be regarded as an approximative expression to those corresponding random equations which had so nice properties from the viewpoint of physics, or engineerings :

$$(* \ 3) \qquad \frac{d}{dt} X(t) = \bar{f}(t, X(t)) + g(t, X(t)) \frac{d}{dt} \tilde{Z}(t,\omega) .$$

where $\{\tilde{Z}(t,\omega) \ t \geq 0\}$ is a random process which is so meaningful from the viewpoint of physics that the expression (* 3) can be understood as a family of deterministic equations, and stands as an approximation process to $Z(t,\omega)$. And where $\bar{f}(t,x)$ is a function which depends on $f(t,x)$ and $g(t,x)$ in (* 1).

Moreover in 1966, he introduced a new type of stochastic integral, called nowadays "the Stratonovich's integral", by which we can regard the equation (* 2) , understanding the stochastic integral in the sense of Stratonovich, as an approximative expression of the following

$$(* 4), \qquad \frac{d}{dt}\tilde{X}(t) = f(t, \tilde{X}(t)) + g(t, \tilde{X}(t))\frac{d}{dt}Z(t,\omega) \quad ,$$

when the quantities $X(t)$, $f(t,x)$, $g(t,x)$ and $Z(t,\omega)$ are assumed to be R^1-valued functions.

On the other hand, the question on the fidelity was also studied by E. Wong and M. Zakai independently. In 1964, they published a series of papers [43], [44], where they considered the relation between the solution of Ito's differential equation and that of the family of random equations :

$$(* 5) \qquad \frac{d}{dt}X^{(n)}(t,\omega) = f(t, X^{(n)}(t,\omega)) + g(t, X^{(n)}(t,\omega))\frac{d}{dt}Z^{(n)}(t,\omega)$$

$$(n = 1, 2, \cdots),$$

where $\{Z^{(n)}(t,\omega),\ t \geq 0\}_{n=1}^{\infty}$ is a family of stochastic processes which converges to the Brownian motion process $Z(t,\omega)$ as n tends to infinity, and each of which has a piecewise smooth sample function.

They studied the convergence of the sequence of stochastic processes $\{X^{(n)}(t,\omega),\ 0 \leq t \leq T\}_{n=1}^{\infty}$ and found that , under certain kind of assumptions on regularities of $f(t,x)$, $g(t,x)$, the sequence converges in the mean to a limit $\{X^*(t,\omega),\ 0 \le t \le T\}$ which is stochastically equivalent to the solution $\{X(t,\omega),\ 0 \le t \le T\}$ of the following Ito's integral equation :

$$(*\ 6), \quad X(t,\omega) - X_0 = \int_0^t \{f(s,\ X(s,\omega)) + \frac{1}{2} gg_x(s,\ X(s,\omega))\}\ ds$$
$$+ \int_0^t g(s,\ X(s,\omega))\ dZ(s,\omega)\ .$$

The equation (* 6) is equivalent to the equation (* 2), if we understand the stochastic integral in (* 2) in the sense of Stratonovich. Thus their result suggests to us that the fidelity of mathematical procedures in such random problems, stated above, is assured as far as we take the stochastic integrals in the sense of Stratonovich.

(Remark) It must be noticed that the above statements on the question of physical fidelity are valid only for the one-dimensional cases, (cf., T. Nakamizo [27]).

As we have explained above, the stochastic integral of Stratonovich serves as a nice tool for the understanding of practical problems. However it has been pointed out that the integral of Stratonovich admits a so narrow class of integrable functions that it is hard to apply to many of interesting problems, (cf., G. Kallianpur- C. Striebel [22]). In fact, we will find it to be inapplicable to the problem treated in Chapter V.

In 1970 the author published a paper on this subject (S. Ogawa [30]), and there he succeeded to introduce the new-types of stochastic integrals which improved away such disadvantage of the integral of Stratonovich. This chapter is devoted to the introduction of the author's new-types of stochastic integrals, which is a full exposition of the paper [30], and a part of [31]. The discussions are developed in the following manner :

In Section 3.2, we will introduce the new-types of stochastic integrals, starting from the considerations on the convergence of a sequence of Riemann sums of a random function. Among these integrals, we will extensively be interested in the integral of index 1/2, since it admits such a remarkable property that it can be understood as a limit of a certain sequence of random Stieltjes integrals. The details on this fact will be explained in Section 3.3. In Section 3.4, we shall compare our integral with the other types of stochastic integrals, namely, the integral of Stratonovich, the integral of Stratonovich - Fisk type which was introduced by Professors G. Kallianpur and C. Striebel in 1971. We shall state some concluding remarks on this subject in the final section, 3.5.

3. 2 The New Types of Stochastic Integrals .

Let $\{Z(t,\omega),\ t\geq 0\}$ be an R^1-valued Brownian motion process and let $\mathcal{N}_t^s$ $(t\geq s\geq 0)$ be the smallest σ-algebra generated by the sets $\{Z(u,\omega) - Z(s,\omega),\ s\leq u\leq t\}$. Let $\mathcal{S}$ be the class of random functions $f(t,\omega)$, defined on $[0,T]\times\Omega$ $(0<T<\infty)$, satisfying the followings ;

$\mathcal{S}$. 1), $f(t,\omega)$ is $\mathcal{B}_{[0,t]}\times\mathcal{N}_t^0$-measurable, where $\mathcal{B}_{[0,t]}$ is the Borel field on the interval $[0,t]$.

$\mathcal{S}$. 2), $E\{\int_0^T f^2(s,\omega)ds\} < \infty$.

We call a family of partitions $\{\Delta^{(n)}\ ;\ 0\leq t_0^{(n)}< t_1^{(n)}<\cdots <t_n^{(n)}\leq T\}$ on the interval $[0,T]$, "canonical" if $\lim_{n\to\infty} \max_{0\leq i\leq n-1} n(t_{i+1}^{(n)} - t_i^{(n)}) < \infty$ and if $\Delta^{(n)}\subset\Delta^{(n+1)}$.

Let us begin with the consideration on the convergence of the following sequence of Riemann sums of a function $f(t,\omega)$ in the class $\mathcal{S}$.

$$J_n^{(k)}(f)(\omega) = \sum_{i=0}^{n-1} f(t_i^{(n)} + k\tau_i^{(n)})\{ Z(t_{i+1}^{(n)},\omega) - Z(t_i^{(n)},\omega)\}\ , \tag{3.1}$$

where $\tau_i^{(n)} = t_{i+1}^{(n)} - t_i^{(n)}$ and k is a nonnegative constant such that $0\leq k\leq 1$.

As for the convergence of the sequence $\{J_n^{(k)}(f)\}_{n=1}^{\infty}$, it is well known that if the constant k is equal to zero, the sequence $\{J_n^{(0)}(f)\}_{n=1}^{\infty}$ converges in the mean to the limit $\mathcal{J}_0^+(f)$, which coincides with the Ito's stochastic integral $\int_0^T f(t,\omega)dZ(t,\omega)$. However for a general case($k \neq 0$), it is hard to know whether the sequence converges or not unless an additional condition is supposed on $f(t,\omega)$. The difficulty of the problem lies in the fact that each random variable $f(t_i^{(n)} + k\tau_i^{(n)})(i = 0, 1, \cdots, n-1)$ is not independent of the corresponding increment $(Z(t_{i+1}^{(n)},\omega) - Z(t_i^{(n)},\omega))$ $(i = 0, 1, \cdots, n-1)$. So it seems to be necessary to set one more condition which describes the way of dependence of $f(t,\omega)$ on the process $\{Z(t,\omega), t \geq 0\}$.

Hence we are led to the following

<u>Definition .</u> We will call that a random function $f(t,\omega)$ belongs to the class $\mathcal{S}^+([0,T])$, if it belongs to the class $\mathcal{S}$ and satisfies the next condition ;

$\mathcal{S}$. 3), $f(t,\omega)$ is uniformly $B^+(M_4)$-differentiable on the interval $[0,T]$ and its B^+-derivative $f(t,\omega)$ is Riemann-integrable on $[o,T]$ in $L^2(\Omega)$-sense.

Now we have

<u>Theorem 3. 1 .</u> For a random function $f(t,\omega)$ in $\mathcal{S}^+([o,T])$, the sequence of Riemann sums $\{J_n^{(k)}(f)\}_{n=1}^{\infty}$ converges in the mean to

the limit $\mathcal{J}_k^+(f)(\omega)$, which satisfies the following relation,

$$\text{(3.2).}\qquad \underset{n\to\infty}{\text{l.i.m}}\; J_n^{(k)}(f) = \mathcal{J}_k^+(f)(\omega) = \mathcal{J}_0^+(f)(\omega) + k\int_0^T \check{f}(t,\omega)dt$$

where $\mathcal{J}_0^+(f)(\omega)$ is the Ito's integral of $f(t,\omega)$ on $[0,T]$ and the second term is the Riemann integral of $\check{f}(t,\omega) = \dfrac{\partial^+}{\partial^+ Z_t} f(t,\omega)$ in $L^2(\Omega)$-sense.

(Proof of Theorem 3.1) Let us put

$$\delta_n(\omega) = J_n^{(k)}(f)(\omega) - J_n^{(0)}(f)(\omega) - \sum_{i=0}^{n-1} \check{f}(t_i^{(n)})(\Delta_i^{(n)}Z)(\Delta_{i,k}^{(n)}Z)$$

where $\Delta_i^{(n)}Z = Z(t_{i+1}^{(n)},\omega) - Z(t_i^{(n)},\omega)$ and $\Delta_{i,k}^{(n)}Z = Z(t_i^{(n)} + k\tau_i^{(n)},\omega) - Z(t_i^{(n)},\omega)$.

From the definition of $J_n^{(k)}(f)(\omega)$, we have

$$\delta_n(\omega) = \sum_{i=0}^{n-1}\{ f(t_i^{(n)} + k\tau_i^{(n)}) - f(t_i^{(n)}) - \check{f}(t_i^{(n)})\Delta_{i,k}^{(n)}Z\}\Delta_i^{(n)}Z .$$

Apllying the Schwarz's inequality, we obtain

$$E\{\delta_n^2(\omega)\} \leq n \sum_{i=0}^{n-1} E\{(f(t_i^{(n)} + k\tau_i^{(n)}) - f(t_i^{(n)}) - \check{f}(t_i^{(n)})\Delta_{i,k}^{(n)}Z)^2 \times (\Delta_i^{(n)}Z)^2\}$$

In fact, we have

$$E\{ (\sum_{i=0}^{n-1} \check{f}(t_i^{(n)}) (\Delta_{i,k}^{(n)} Z \cdot \Delta_i^{(n)} Z - k\tau_i^{(n)}))^2 \}$$

$$= \sum_{i=0}^{n-1} E\{ \check{f}^2(t_i^{(n)}) E\{ (\Delta_{i,k}^{(n)} Z \cdot \Delta_i^{(n)} Z - k\tau_i^{(n)})^2 | \mathcal{N}^0_{t_i^{(n)}} \} \}$$

$$+ 2 \sum_{i\ j}^{n-1} E\{ \check{f}(t_i^{(n)}) \check{f}(t_j^{(n)}) (\Delta_{j,k}^{(n)} Z \cdot \Delta_j^{(n)} Z - k\tau_j^{(n)})$$

$$\times \ E\{ (\Delta_{i,k}^{(n)} Z \cdot \Delta_i^{(n)} Z - k\tau_i^{(n)}) | \mathcal{N}^0_{t_i^{(n)}} \} \}$$

$$= \sum_{i=0}^{n-1} E\{ \check{f}^2(t_i^{(n)}) E\{ (\Delta_{i,k}^{(n)} Z \cdot \Delta_i^{(n)} Z - k\tau_i^{(n)})^2 | \mathcal{N}^0_{t_i^{(n)}} \} \}$$

$$= \sum_{i=0}^{n-1} E\{ \check{f}^2(t_i^{(n)}) \} (k^2 + k) (\tau_i^{(n)})^2 \leq (k^2+k) MT \cdot \max_{0 \leq i \leq n-1} (\tau_i^{(n)})$$

$$(M = \sup E\{ \check{f}^2(t,\omega) \} < \infty) ,$$

which means that the equality in (3.4) holds.

The assertion of Theorem is a direct consequence of equations (3.3), (3.4) and the estimate below,

$$E\{ (J_n^{(k)}(f)(\omega) - \mathcal{J}_0^+(f)(\omega) - k \int_0^T \check{f}(t,\omega) dt)^2 \}$$

$$\leq 3E\{\delta_n^2(\omega)\} + 3E\{(J_n^{(0)}(f)(\omega) - \mathcal{J}_0^+(f)(\omega))^2\}$$

$$+ 3E\{\{ \sum_{i=0}^{n-1} \check{f}(t_i^{(n)})\Delta_{i,k}^{(n)}Z\cdot\Delta_i^{(n)}Z - k \int_0^T \check{f}(t,\omega)dt\}^2\} \quad .$$

Thus we have obtained the family of stochastic integrals $\{\mathcal{J}_k^+(f)(\omega),\ 0\leq k\leq 1\}$ for functions in $\mathcal{S}^+([0,T])$. Hereafter we will call $\mathcal{J}_k^+(f)(\omega)$, " the stochastic integral of index k ".

It is clear how (3.2) yields the definition of the stochastic integral of index k over any subinterval [s,t] in [0,T]. And it is also clear that the stochastic integral of index k has the usual properties as an integration. Namely, we have the following assertion which is easy to verify.

Proposition 3. 1. Let $f(t,\omega)$ and $g(t,\omega)$ be functions in the class $\mathcal{S}^+([0,T])$. Then we have for any constants C_1, C_2 the next

$$(3.5),\quad \mathcal{J}_k^+(C_1f + C_2g) = C_1\mathcal{J}_k^+(f) + C_2\mathcal{J}_k^+(g) ,$$

and for any point u in [s,t] ($\subset$[0,T]),

$$(3.6)\quad \int_s^t f(v,\omega)d^{(k)}Z(v,\omega) = \int_s^u f(v,\omega)d^{(k)}Z(v,\omega) + \int_u^t f(v,\omega)d^{(k)}Z(v,\omega)$$

, where the notation $\int d^{(k)}Z(v,\omega)$ means the stochastic integral of index k.

For an arbitrary function $f(t,\omega)$ in $\mathcal{S}^{+}([0,T])$, let us put

$$(3.7) \qquad F(t,\omega) = \int_0^t f(s,\omega)d^{(k)}Z(s,\omega), \qquad (t \leq T).$$

Then our theorem tells us that the stochastic process $\{F(t,\omega), 0 \leq t \leq T\}$ is almost surely continuous in t and posseses the derivatives $DF(t,\omega)$, $\frac{\partial^{+}}{\partial^{+} Z_t} F(t,\omega)$. Moreover, we know that the derivatives satisfy the follwoing

$$(3.8), \qquad DF(t,\omega) = k \frac{\partial^{+}}{\partial^{+} Z_t} \left(\frac{\partial^{+}}{\partial^{+} Z_t} F(t,\omega) \right) .$$

On the other hand, we can prove that the condition above completely characterizes those stochastic processes that are represented in the form of the stochastic integral of index k .

Proposition 3. 2. If a stochastic process $\{X(t,\omega), 0 \leq t\}$ is M-differentiable and twice uniformly $B^{+}(M_4)$-differentiable on an interval $[0,a]$. And if the derivatives satisfy (3.8) for any t in $[0,a]$, then tere exists a stochastic process $\{\tilde{X}(t,\omega), 0 \leq t\}$ which belongs to the class $\mathcal{S}^{+}([0,T])$ and satisfies the next

$$(3.9). \qquad X(t,\omega) - X(0,\omega) = \int_0^t \tilde{X}(s,\omega)d^{(k)}Z(s,\omega), \quad \text{a. s. for each } t.$$

(Proof of Proposition 3.2). For the proof, we notice that the assertion of Proposition 2.8 also holds for such stochastic processes whose M-derivatives are Riemann integrable in $L^1(\Omega)$-sense.

(In fact we did not make use of the continuity of M-derivatives in the proof of Proposition 2.8, except for yielding the property that the derivative is Riemann integrable in $L^1(\Omega)$-sense.)

Because $\check{X}(t,\omega)$ belongs to $\mathcal{S}^+([0,T])$ and thus $\frac{\partial^+}{\partial^+ Z_t} \check{X}(t,\omega)$ is Riemann integrable on $[0,a]$ in $L^2(\Omega)$-sense, the derivative $DX(t,\omega)$ is also Riemann integrable on $[0,a]$ in $L^1(\Omega)$-sense by virtue of (3.8). Hence we know from Proposition 2.8 and the remark stated above, the validity of the following

$$X(t,\omega) - X(0,\omega) = \int_0^t DX(s,\omega)ds + \int_0^t \check{X}(s,\omega)d^{(0)}Z(s,\omega) \quad .$$

This completes the proof, since we have $DX(t,\omega) = k \frac{\partial^+}{\partial^+ Z_t} \check{X}(t,\omega)$.

3. 3 The Stochastic Integral $\mathscr{I}^{+}_{1/2}$.

In the previous section, we have obtained the family of stochastic integrals $\{\mathscr{I}^{+}_{k}(f),\ 0\leq k\leq 1\}$ which includes the Ito's integral as its special case. Among these, we are extensively interested in the stochastic integral of index $1/2$, because it has a few nice properties from the viewpoint of applied mathematics, as we shall see from now on.

Let $\{ \Delta^{(n)}\}$ be a canonical family of partitions on the interval $[0,T]$ and let $\{Z^{(n)}(t,\omega),\ 0\leq t\leq T\}_{n=1}^{\infty}$ be the family of stochastic processes defined in the follwoing manner :

$$(3.10) \qquad Z^{(n)}(t,\omega) = Z(t_i^{(n)},\omega) + \frac{\Delta_i^{(n)}Z}{\tau_i^{(n)}}(t - t_i^{(n)}) \ , \text{ for } t_i^{(n)} \leq t \leq t_{i+1}^{(n)},$$

$$\text{where } \tau_i^{(n)} = t_{i+1}^{(n)} - t_i^{(n)} \ , \quad \Delta_i^{(n)}Z = Z(t_{i+1}^{(n)},\omega) - Z(t_i^{(n)},\omega)$$

$$\text{and } \{t_i^{(n)}\}_{i=1}^{n} \in \Delta^{(n)} \ .$$

Then it is easy to verify that each stochastic process $\{Z^{(n)}(t,\omega),\ 0\leq t\leq T\}$ has an almost surely piecewise smooth sample function and that the sequence $\{Z^{(n)}\}$ converges in the mean, uniformly in t, to the Brownian motion process Z. That is, we have

$$(3.11). \qquad \lim_{n\to\infty}\ \sup_{0\leq t\leq T}\ E\{(Z(t,\omega) - Z^{(n)}(t,\omega))^2\} = 0 \ .$$

Let $\{f^{(n)}(t,\omega)\}_{n=1}^{\infty}$ be a family of random functions each of which is defined on $\Omega\times[0,T]$ and satisfies the following conditions ;

f.1) $f^{(n)}(t,\omega)$ is $\mathcal{B}_{[0,t]}\times\mathcal{N}^{0}_{[t]}$ -measurable, where $[t] = t^{(n)}_{i+1}$ when $t^{(n)}_{i} \leq t < t^{(n)}_{i+1}$, $(i = 0, 1, 2, \cdots, n-1)$.

f.2) There exists a function $f(t,\omega)$ in $\mathcal{S}^{+}([0,T])$, such that

$$\lim_{n\to\infty}\sup_{0\leq t\leq T} E\{(f(t,\omega) - f^{(n)}(t,\omega))^2\} = 0 .$$

We may regard the stochastic process $\{Z^{(n)}(t,\omega),\ 0\leq t\leq T\}$ as a physically meaningful modification of the Brownian motion process $\{Z(t,\omega),\ t\geq 0\}$ since we know that the family has a so nice properties mentioned before. The random functions are therefore understood as those which might be expected to appear in the stochastic problems when we represent random phenomena using the stochastic process $\{Z^{(n)}(t,\omega),\ 0\leq t\leq T\}$ as a physically natural modification of the Brownian motion.

In this chapter, we are interested in the question of the convergence of the sequence of random Stieltjes integrals $\{S(f^{(n)})\}_{n=1}^{\infty}$ each of which is defined as follows ;

$$(3.12),\qquad S(f^{(n)})(\omega) = \int_0^T f^{(n)}(t,\omega)dZ^{(n)}(t,\omega)$$

$$= \sum_{i=0}^{n-1} \frac{\Delta^{(n)}_i Z}{\tau^{(n)}_i} \int_{t^{(n)}_i}^{t^{(n)}_{i+1}} f^{(n)}(t,\omega)dt \quad ,$$

and we are also interested in the relation between the integral $\mathcal{J}^{+}_{1/2}(f)$ and the limit of $\{S(f^{(n)})\}_{n=1}^{\infty}$ if it exists.

We will call that a family of random functions $\{f^{(n)}(t,\omega)\}_{n=1}^{\infty}$ has the property α, if there exists a sequence of random functions $\{\tilde{f}^{(n)}(t,\omega)\}_{n=1}^{\infty}$ which satisfies the following conditions ;

$(\alpha,1)$, $\quad \tilde{f}^{(n)}(t,\omega)$ is $\mathcal{B}_{[0,t]}\times\mathcal{N}^{0}_{[t]}$ -measurable .

$(\alpha,2)$, $\quad E\{(\tilde{f}^{(n)}(t,\omega))^4\} < \infty$, for any $t(\geq 0)$.

$$(\alpha,3), \quad \lim_{n\to\infty} \max_{p\in\Delta^{(n)}} \sup_{p'\in[p,p(n)]} E\{\frac{1}{\sqrt{\tau'_p}}\{f^{(n)}(p') - f^{(n)}(p) - \tilde{f}^{(n)}(p) \times \Delta^{(n)}_{p',p}Z\}\}^4 = 0 ,$$

where $p(n)$ is the nearest right point of p in $\Delta^{(n)}$ and $\tau'_p = p' - p$, $\Delta^{(n)}_{p',p}Z = Z^{(n)}(p',\omega) - Z^{(n)}(p,\omega)$.

Here we must notice that a sequence of such random functions $\{\tilde{f}^{(n)}(t,\omega)\}_{n=1}^{\infty}$ may not be uniquely determined. However the limit of the sequence $\{\tilde{f}^{(n)}(p,\omega)\}_{n=1}^{\infty}$ is determined uniquely for any fixed p in $\Delta^{(n)}$, if it exists. In fact, if we let $\{\tilde{g}^{(n)}(p,\omega)\}_{n=1}^{\infty}$ be another sequence at point p, then we have

$$E\{(\tilde{f}^{(n)}(p,\omega) - \tilde{g}^{(n)}(p,\omega))^2\} = E\{\frac{1}{\sqrt{\tau_p^{(n)}}}(\tilde{f}^{(n)}(p,\omega) - \tilde{g}^{(n)}(p,\omega))\Delta_p^{(n)}Z^{(n)}\}^2$$

$$\leq 2E\{\frac{1}{\sqrt{\tau_p^{(n)}}}(f^{(n)}(p(n)) - f^{(n)}(p) - \tilde{f}^{(n)}(p)\Delta_p^{(n)}Z)^2\}$$

$$+ 2E\{\frac{1}{\sqrt{\tau_p^{(n)}}}(f^{(n)}(p(n)) - f^{(n)}(p) - \tilde{g}^{(n)}(p)\Delta_p^{(n)}Z)^2\}$$

where $\tau_p^{(n)} = p(n) - p$ and $\Delta_p^{(n)}Z = Z^{(n)}(p(n),\omega) - Z^{(n)}(p,\omega)$.

From the estimate above and the condition $(\alpha,3)$, we conclude the uniqueness of the limit .

<u>Theorem 3.2</u> Let $\{f^{(n)}(t,\omega)\}_{n=1}^{\infty}$ be a family of random functions satisfying conditions f.1), f.2) and the following

f,3) ; The family $\{f^{(n)}(t,\omega)\}_{n=1}^{\infty}$ has the property α, and each component of random vectors $\{\{f^{(n)}(p,\omega),\ p\in\Delta^{(n)}\}\}_{n=1}^{\infty}$ satisfies the relation ; $\lim_{n\to\infty} \max_{p\in\Delta^{(n)}} E\{(f^{(n)}(p,\omega) - \check{f}(p,\omega))^2\} = 0$, where $\check{f}(p,\omega) = \frac{\partial^+}{\partial^+ Z_t} f(t,\omega)\Big|_{t=p}$ and $f(t,\omega)$ is the function just described in the condition f.2).

Then we have

$$(3.13),\qquad \underset{n\to\infty}{\mathrm{l.i.m}}\ S(f^{(n)})(\omega) = \mathcal{J}^{+}_{1/2}(f) \quad .$$

For the proof of this theorem, it is necessary to check the following assertion.

Lemma ,

$$\underset{n\to\infty}{\mathrm{l.i.m}} \sum_{i=0}^{n-1} \frac{\Delta_i^{(n)}Z}{\tau_i^{(n)}} \tilde{f}^{(n)}(t_i^{(n)}) \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} (Z^{(n)}(t,\omega) - Z^{(n)}(t_i^{(n)},\omega))dt$$

$$= \frac{1}{2} \int_0^T \frac{\partial^+}{\partial^+ Z_t} \check{f}(t,\omega)\, dt \quad ,$$

where $\{f^{(n)}(t,\omega)\}_{n=1}^{\infty}$ is a sequence of random functions mentioned in f.3) .

(Proof of Lemma) It is not difficult to check the following

$$(3.14), \qquad \underset{n\to\infty}{\mathrm{l.i.m}} \sum_{i=0}^{n-1} \check{f}(t_i^{(n)},\omega)(\Delta_i^{(n)}Z)^2 = \int_0^T \check{f}(t,\omega)dt \quad ,$$

where $\check{f}(t,\omega) = \dfrac{\partial^+}{\partial^+ Z_t} f(t,\omega)$.

Since $Z^{(n)}(t,\omega) - Z^{(n)}(t_i^{(n)},\omega) = \dfrac{\Delta_i^{(n)}Z}{\tau_i^{(n)}} (t - t_i^{(n)})$ for

any t in $[t_i^{(n)}, t_{i+1}^{(n)})$, we have

$$\delta_n(\omega) = \sum_{i=0}^{n-1} \{\frac{\Delta_i^{(n)}Z}{\tau_i^{(n)}} \tilde{f}^{(n)}(t_i^{(n)}) \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} (Z^{(n)}(t,\omega) - Z^{(n)}(t_i^{(n)},\omega))dt$$

$$- \frac{1}{2} \check{f}(t_i^{(n)})(\Delta_i^{(n)}Z)^2 \}$$

$$= \frac{1}{2} \sum_{i=0}^{n-1} \{\tilde{f}^{(n)}(t_i^{(n)}) - \check{f}(t_i^{(n)})\}(\Delta_i^{(n)}Z)^2 \quad .$$

Applying the Schwarz's inequality, we get

$$E\{(\delta_n(\omega))^2\} \leq \frac{n}{4}\sum_{i=0}^{n-1} E\{(\tilde{f}^{(n)}(t_i^{(n)}) - \check{f}(t_i^{(n)}))^2(\Delta_i^{(n)}Z)^4\}$$

$$\leq \frac{3}{4} n(\max_{0\leq i\leq n-1} \tau_i^{(n)}) \sum_{i=0}^{n-1} E\{(\tilde{f}^{(n)}(t_i^{(n)}) - \check{f}(t_i^{(n)}))^2\}\tau_i^{(n)}.$$

From this estimate, the relation (3.14) and the condition f.3), we get the conclusion of Lemma.

(Proof of Theorem 3.2) Let us put

$$\eta_n(\omega) = S(f^{(n)}) - \sum_{i=0}^{n-1}(\Delta_i^{(n)}Z)\{f^{(n)}(t_i^{(n)}) - \frac{\tilde{f}^{(n)}(t_i^{(n)})}{\tau_i^{(n)}} \times \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} (Z^{(n)}(t,\omega) - Z^{(n)}(t_i^{(n)},\omega))dt\}$$

$$= \sum_{i=0}^{n-1} \frac{\Delta_i^{(n)}Z}{\tau_i^{(n)}} \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} \{f^{(n)}(t) - f^{(n)}(t_i^{(n)}) - \tilde{f}^{(n)}(t_i^{(n)}) \times (Z^{(n)}(t,\omega) - Z^{(n)}(t_i^{(n)},\omega))\}dt .$$

Then we have,

$$E\{(\eta_n(\omega))^2\} \leq n\sum_{i=0}^{n-1}(3\tau_i^{(n)})^{\frac{1}{2}}\{\int_{t_i^{(n)}}^{t_{i+1}^{(n)}} E\{f^{(n)}(t) - f^{(n)}(t_i^{(n)}) - \tilde{f}^{(n)}(t_i^{(n)}) \times (Z^{(n)}(t,\omega) - Z^{(n)}(t_i^{(n)},\omega))\}^4 dt\}^{\frac{1}{2}} .$$

For a sufficiently large n, we have the following estimate.

$$E\{f^{(n)}(t) - f^{(n)}(t_i^{(n)}) - \tilde{f}^{(n)}(t_i^{(n)})(Z^{(n)}(t,\omega) - Z^{(n)}(t_i^{(n)},\omega))\}^4$$

$$\leq (t - t_i^{(n)})\cdot \varepsilon(\tau_i^{(n)}) \quad ,$$

for any fixed t in $[t_i^{(n)}, t_{i+1}^{(n)})$, by virtue of condition f.3), where $\varepsilon(\cdot)$ is a nonnegative function such that $\lim_{n\to\infty} \varepsilon(\tau_i^{(n)}) = 0$.

Hence,

$$(3.15) \qquad E\{(\eta_n(\omega))^2\} \leq n \sum_{i=0}^{n-1} (\tau_i^{(n)})^2 \cdot \varepsilon^{\frac{1}{2}}(\tau_i^{(n)})$$

$$\leq n\cdot \max_{n\leq i\leq n-1} \tau_i^{(n)} \sum_{i=0}^{n-1} \varepsilon^{\frac{1}{2}}(\tau_i^{(n)})\tau_i^{(n)} \quad .$$

On the other hand, we have the following

$$(3.16). \qquad E\{(S(f^{(n)}) - \mathcal{J}_0^+(f) - \frac{1}{2}\int_0^T \check{f}(t,\omega)dt \;)^2\}$$

$$\leq 3E\{(\eta_n(\omega))^2\} + 3E\{(\sum_{i=0}^{n-1} f^{(n)}(t_i^{(n)})\Delta_i^{(n)} Z - \mathcal{J}_0^+(f) \;)^2\}$$

$$+ 3E\{(\sum_{i=0}^{n-1} \frac{\Delta_i^{(n)} Z}{\tau_i^{(n)}} \tilde{f}^{(n)}(t_i^{(n)}) \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} (Z^{(n)}(t,\omega) - Z^{(n)}(t_i^{(n)},\omega))dt$$

$$- \frac{1}{2}\int_0^T \check{f}(t,\omega)dt \;)^2\} \quad .$$

The conclusion of the theorem follows from the estimate above, (3.16) (3.15) and Lemma, since we have

$$\lim_{n\to\infty} E\{(\sum_{i=0}^{n-1} f^{(n)}(t_i^{(n)})\Delta_i^{(n)}Z - \mathcal{J}_0^+(f))^2\}$$

$$\leq 2 \lim_{n\to\infty} E\{(\sum_{i=0}^{n-1} (f^{(n)}(t_i^{(n)}) - f(t_i^{(n)}))\Delta_i^{(n)}Z)^2\}$$

$$+ 2 \lim_{n\to\infty} E\{(\sum_{i=0}^{n-1} f(t_i^{(n)})\Delta_i^{(n)}Z - \mathcal{J}_0^+(f))^2\}$$

$$= 0 \quad ,$$

by virtue of the condition f.2). Q.E.D

(Remark). We have proved the assertion of the theorem under the condition f.2). However the discussions in the proof tells us that the assertion also holds under a slight weaker condition as follows ;

f.2)', There exists a function $f(t,\omega)$ in $\mathcal{S}^+([0,T])$, such that

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} E\{(f^{(n)}(t_i^{(n)}) - f(t_i^{(n)}))^2\}\tau_i^{(n)} = 0.$$

3. 4, Comparisons with Other Types of Stochastic Integrals.

In the previous section, we have introduced a family of stochastic integrals $\{\mathcal{J}^{+}_{k}(f),\ 0\leq k\leq 1\}$, and established the theorem stating that the integral $\mathcal{J}^{+}_{1/2}(f)$ can be understood as a limit of a sequence of random Stieltjes integrals. In this section, we compare the integral of index $1/2$ with other types of stochastic integrals.

For a continuous function $f(t,x)$ which has a continuous derivative $f_x(t,x)$, Stratonovich studied the convergence of the following Riemann sums ;

$$(3.17)\qquad R_n(f) = \sum_{i=0}^{n-1} f[t_i^{(n)}, \frac{1}{2}(X(t_i^{(n)}) + X(t_{i+1}^{(n)}))]\Delta_i^{(n)}Z \quad , \qquad (n = 1,2,\cdots)$$

where $\{\Delta^{(n)};\ 0\leq t_0^{(n)} < t_1^{(n)} < \cdots < t_n^{(n)} \leq T\}_{n=1}^{\infty}$ is a family of partitions on $[0,T]$ providing $\lim_{n\to\infty}\max_{0\leq i\leq n-1}(t_{i+1}^{(n)} - t_i^{(n)}) = 0$,

and where $\{X(t,\omega),\ t\geq 0\}$ is a classical diffusion process defined by

$$(3.18),\qquad dX(t,\omega) = a(t,\ X(t,\omega))dt + b(t,\ X(t,\omega))dZ(t,\omega)\ .$$

As a result of this consideration, he reached the following

Theorem (R. L. Stratonovich). If $a(t,x)$ and $b(t,x)$ are continuous functions, and if $f(t,x)$ admits the following condition ;

$$\int_0^T E\{f^2(t, X(t))b^2(t, X(t))\}dt < \infty ,$$

then the sequence $\{R_n(f)\}$ converges in the mean to the following limit,

$$(3.19). \quad \underset{n \to \infty}{\mathrm{l.i.m}}\, R_n(f) = \mathscr{J}_0^+(f) + \frac{1}{2}\int_0^T f_x(t, X(t))b(t, X(t))dt .$$

On the base of this fact, he introduced the new type of stochastic integral which he called "a symmetrized integral", in the following way ;

$$(3.20), \quad (S)\text{-}\int_0^T f(t,X(t))dZ(t,\omega) = (I)\text{-}\int_0^T f(t,X(t))dZ(t,\omega)$$
$$+ \frac{1}{2}\int_0^T f_x(t,X(t))b(t,X(t))dt .$$

(Remark) The notations (S)-, (I)-, mean that the integrals are taken in the sense of Stratonovich, or Ito respectively.

For a class of such random functions that are integrable in the sense of Stratonovich, it is easy to check that they are also integrable in our sense when they satisfy

$$(3.21)_{(i)}, \quad E\{f^4(t,X(t))\} < \infty \quad (0 \leq t \leq T)$$

$$(3.21)_{(ii)}, \quad |f(t,x) - f(s,x)| \leq C(x)|s - t|^{\gamma} \left(\gamma > \frac{1}{2}\right) ,$$

where $C(x)$ is such that $\int_0^T E\{C^4(X(t))\}dt < \infty$.

Moreover we can see that for such a random function both intagrals coincide with each other, since the random function $f(t, X(t))$ is uniformly $B^{+}(M_4)$-differentiable on $[0,T]$ with $f_x(t,X(t))b(t,X(t))$ its B^{+}-derivative. In other words, if we restrict our attentions to such random functions that are represented in the form $f(t,X(t))$ our integral has a less wide class of integrable functions than that of the Stratonovich. However our integral is applicable to a more general types of random functions than the Stratonovich's. Namely the integrable class of functions in our case is not restricted in the classical diffusion processes while the integrable class is so in the Stratonovich's case.

After the author's publication on this subject, Professors G. Kallianpur and C. Striebel proposed an analogous idea of defining the stochastic integrals, which was a direct application of Fisk's results, (D. L. Fisk [12]). Let us give a short sketch of their integral following Kallianpur-Striebel [23] .

For continuous quasi-martingales $\{N(t),\mathcal{F}_t\}$, $\{M(t),\mathcal{F}_t\}$ on a probability space $(\Omega,\mathcal{F}, P)$,($\{\mathcal{F}_t\}_{t\geq 0}$ is an increasing family of σ-fields each of which is included in $\mathcal{F}$), D. L. Fisk studied the convergence of the following limit ;

$$(3.21), \quad (F)\text{-}\int_0^T N(t)dM(t) = \frac{1}{2}\, \underset{n\to\infty}{\text{p-lim}} \sum_{i=0}^{n-1} \{N(t_{i+1}^{(n)}) + N(t_i^{(n)})\} \times \{M(t_{i+1}^{(n)}) - M(t_i^{(n)})\} ,$$

and obtained the next

Theorem. (D. L. Fisk) The limit in (3,21) exists and satisfies

$$(3.22), \qquad (F)-\int_0^T N(t)dM(t) = (I)-\int_0^T N(t)dM(t)$$

$$+ \frac{1}{2} \text{p-}\lim_{n\to\infty} \sum_{i=0}^{n-1} \Delta_i^{(n)} N_1 \Delta_i^{(n)} M_1,$$

where $\Delta_i^{(n)} N_1 = N_1(t_{i+1}^{(n)}) - N_1(t_i^{(n)})$, $\Delta_i^{(n)} M_1 = M_1(t_{i+1}^{(n)}) - M_1(t_i^{(n)})$

and $N_1(t)$, $M_1(t)$ are martingale-parts of the corresponding quasi-martingales, (cf., D. L. Fisk [12]).

Kallianpur and Striebel applied this result to the following quasi-martingales

$$(3.23), \qquad dM(t) = a_1(t)dt + b_1(t)dZ(t), \quad dN(t) = a_2(t)dt + b_2(t)dZ(t) ,$$

where $a_i(t)$, $b_i(t)$ (i=1, 2) are such that

$$(3.24) \qquad \int_0^T E|a_i(t)|dt , \int_0^T E\{b_i^2(t)\}dt < \infty \quad (i=1,2) ,$$

and they reached the following

Theorem. (Kallianpur- Striebel)

$$(3.25) \qquad (F)-\int_0^T N(t)dM(t) = (I)-\int_0^T N(t)dM(t) + \frac{1}{2}\int_0^T b_1(t)b_2(t)dt .$$

They called this the "Stratonovich-Fisk integral", and applied it to a problem on nonlinear filterings of stochastic processes. Let us compare this with our integral $\mathscr{I}^{+}_{1/2}(f)$. For a convenience of discussions, we consider the case where $dM(t) = dZ(t)$. Then we have from (3.25), the following

$$(3.25)', \quad (F)\text{-}\int_0^T N(t)dZ(t) = (I)\text{-}\int_0^T N(t)dZ(t) + \frac{1}{2}\int_0^T b_2(t)dt ,$$

which coincides with our integral when $b_2(t)$ is continuous in $L^4(\Omega)$-sense. On the other hand, it is obvious that there is a stochastic process which is integrable in our sense but not in the sense of Stratonovich-Fisk, because the uniformly $B^+(M_4)$-differentiable processes do not necessarily have the Ito's differential formula. In other words, the integrable class is not restricted to quasi-martingales in our case. As for the relation between the in-tegral of Stratonovich and that of Stratonovich-Fisk, it is also clear that they coincide with each other when the random function $f(t,X(t))$ admits the Ito's differential formula, (i.e., the function is continuous in t and has a continuous derivative $f_{xx}(t,x)$).

Summing up the discussions in this section, we can say that the degrees of usefulness of these three types of integrals depend on the individual situations of problems to which they are applied.

3. 5 Concluding Remarks.

In this chapter we have introduced the new types of stochastic integrals and investigated their properties in some details. Among the assertions established here, Theorem 3.2 is the most important one. The theory of stochastic differential equations has its own manner of considerations which is quite different from those in other deterministic theories such as the theory of partial differential equations, of ordinary differential equations. The author thinks that it is this difference which prevented those deterministic theories from the direct application to the random problems that are concerned with the Brownian motion, or in return, prevented the theory of stochastic differential equations from the vivid applications to the practical random problems. At this point , the author expects the theorem 3.2 to perform as a bridge between the stochastic theory and those deterministic ones, since it establishes the relation between the stochastic integral and the sequence of random Stieltjes integrals each of which admits a physical interpretation.

In the course of discussions, we have seen that the notion of B^+-derivatives acted an important role. Following the same considerations in this chapter, it is natural to expect that the notion of B^--derivatives will be used in the definition of the "time-reversed integral". We finish this chapter with a comment on this subject.

Let $f(t,\omega)$ be an R^1-valued random function which is $\mathcal{B}_{[t,T]} \times \mathcal{N}_T^t$-measurable and satisfies $\int_0^T E\{f^2(t,\omega)\}dt < \infty$.

Then the definition of the Ito's integral can not apply to such a random function. However it is not difficult to verify the convergence in the mean of the sequence $\{S_n(f)\}$, each element of which is given as follows ;

$$S_n(f) = \sum_{i=0}^{n-1} f(t_{i+1}^{(n)})\Delta_i^{(n)} Z \ , \quad (n = 1, 2, \cdots).$$

We denote this limit as $\bar{\mathcal{J}}_0(f)$ and call " the time-reversed integral", because we can consider, with the definition of stochastic integral, the stochastic equation

$$-X(t,\omega) + x = \int_t^T a(s,X(s,\omega))ds + \int_t^T b(s,X(s,\omega))d^- Z(s,\omega) \ ,$$

which may represent a backward development in time of a diffusion process.

As a modified form of the time-reversed integral $\bar{\mathcal{J}}_0(f)$, we can define the integral $\bar{\mathcal{J}}_k(f)$ $(0 \leq k \leq 1)$,as a limit in the mean of the following Riemann sums,

$$S_n^{(k)}(f) = \sum_{i=0}^{n-1} f(t_{i+1}^{(n)} - k\tau_i^{(n)})\Delta_i^{(n)} Z \quad .$$

In fact it is not difficult to verify the convergence of the sequence $\{S_n^{(k)}(f)\}_{n=1}^{\infty}$ when the function $f(t,\omega)$ is uniformly $B^-(M_4)$-differentiable on $[0,T]$ with a derivative, which is Riemann integrable in $L^2(\Omega)$-sense. Following a similar consideration in this chapter we may find that the limit of the sequence, $\mathscr{I}_k^-(f)$ satisfies the relation ;

$$\mathscr{I}_k^-(f) = \mathscr{I}_0^-(f) + k \int_0^T \check{f}(t,\omega)dt \ , \quad \check{f}(t,\omega) = \frac{\partial^-}{\partial^- Z_t} f(t,\omega).$$

CHAPTER IV Wave Propagation in Random Media (I).

" Wave Propagation in the Presence of the White Noise. "

4. 1. Introduction.

In this chapter we are concerned with the statistical properties of random wave motions that propagate in transmission media into which the "White noise" enters as an external disturbance. The problem is stimulated by recent remarkable progresses in such theories of stochastic systems as the theory of stochastic controls, the statistical theory of communications, etc., where it is desired to analyze behaviours of some practical systems with random disturbances. The stochastic process "white noise" may be one of the most popular mathematical concept which is employed to represent external disturbances, especially in theories where one works with the concept of "observations of signals in the presence of noises". When a stochastic process is introduced to represent a random external disturbace, it seems to be customary in those theories to employ the following hypothesis as a convenient starting point for discussions ;

Hypothesis of the additive noise : A signal (or any physical quantity in a practical system,) is observed as the sum of a certain noise and the original signal which is transmitted.

Therefore it may be important, and interesting as well, to investigate the statistical properties of a stochastic process which is observed as an additional noise to the pure signal as a result of the presence of the white noise in the transmission medium. Problems of such kinds can be formulated as those of wave propagation in random media. The wave motion in a randomly disturbed medium is represented mathematically as the random function which satisfies a partial differential equation with a random coefficient. Hence in this chapter we are concerned with an initial value problem of the partial differential equation with the white noise as an external force term. Moreover we will restrict our attentions to an equation of the first order for the brevity of considerations.

Hereafter we will give a precise formulation of the problem in the next section, 4.2. In section 4.3, we will establish theorems of the existence and the uniqueness of solutions. There we will find that a unique solution is given in an explicit form. And in section 4.4, we shall give a precise definition of "the additive noise of a linear system" which may be different from that is employed in the usual theory of stochastic communications but is a convenient definition from the viewpoint of mathematics, and we will discuss its statistical properties with an example. "The additive noise of a linear system" is a stochastic process which is inherent in the linear system and propagates following the same law by which the pure signal propagates and grows up to be a random wave motion. We will finish discussions with some concluding remarks which will be stated in the final section, 4.5.

4. 2 Formulation of The Problem.

Generally speaking, a transmission medium through which a signal propagates is characterized mathematically by a partial differential equation of hyperbolic type. Therefore the wave motion which propagates in such a medium that is disturbed by the white noise, can be represented as the random function that satisfies a partial differential equation with the white noise as an external force term. For the brevity of considerations, we restrict ourselves to the stochastic partial differential equation of the first order.

Hence our problem may be reduced to the initial value problem of the following formal equation ;

$$(4.\ 1)\qquad \frac{\partial}{\partial t} u(t,x;\,\omega) + a(t,x)\frac{\partial}{\partial x} u(t,x;\,\omega) = A(t,x)u(t,x;\,\omega) + b(t,x)\frac{d}{dt} Z(t,\,\omega)$$

, with the following initial data

$$(4.\ 2),\qquad u(0,x;\,\omega) = u_0(x)\quad ,$$

where $Z(t,\,\omega)$ is a standard Brownian motion process and $\frac{d}{dt}Z(t,\,\omega)$ stands for its formal derivative, namely, the so called "white noise".

Since the white noise is such a stochastic process that does not have ordinary functions as its sample path and therefore must be understood in a sense of random distributions (cf., K. Ito [19] or Gel'fand-Vilenkin [13]), the formal equation (4.1) must be interpreted as the equation of random distributions, while equations treated in the theroy of wave propagation in random media, usualy have a precise meaning for each fixed parameter.

On the other hand, when we use the white noise in problems of applied mathematics, it is customary to recognize it as a mathematical idealization for some physical quantities. Therefore also at this time we wish to consider the equation as a mathematical model for those phenomena which are to be represented by the following type of equations;

$$(4.3) \qquad \frac{\partial}{\partial t} u^{(n)}(t,x;\ \omega) + a(t,x)\,\frac{\partial}{\partial x}\, u^{(n)}(t,x;\ \omega)$$

$$= A(t,x)u^{(n)}(t,x;\ \omega) + b(t,x)\,\frac{d}{dt} Z^{(n)}(t,\ \omega)$$

where $\{Z^{(n)}(t,\omega),\ t\geq 0\}_{n=1}^{\infty}$ is a family of random functions each element of which has some nice properties from the viewpoint of physics (e.g., the almost sure smoothness, etc.), and converges in a certain sense to the Brownian motion process $Z(t,\omega)$ as n tends to infinity.

Hence for a setting of a natural definition of solutions of equation (4.1), it may be desirable that solutions of (4.1) could be understood as a limit of those approximate random functions $u^{(n)}(t,x;\omega)$ which satisfy equations like (4.3). Keeping this in mind, we give the definition of solutions by the following.

Definition   A random function $u(t,x;\omega)$ which is defined on $\Omega \times G_{[0,T]}$ ($G_{[0,T]}$ is the slab $\{(t,x);\ t\in[0,T],\ x\in R^1\}$), is called the solution of the intial value problem (2.1), (2.2), if it satsfies following conditions;

S. 1) $u(t,x;\omega)$ is a function which is continuous in each t and x with respect to the $L^2(\Omega)$-norm, and is in the class $L^2_{loc}(\Omega \times G_{[0,T]})$.
That is, for any compact set M included in $G_{[0,T]}$, $u(t,x;\omega)$ satisfies the next

$$E\{\int_M u^2(t,x;\omega)dtdx\} < \infty \quad .$$

S. 2)   For any continuously differentiable function $v(t,x)$ which has a compact support in $G_{[0,T]}$, $u(t,x;\omega)$ satisfies the following equation with probability one,

$$\int_{G[0,T]} vL(u)dtdx = \int_{G_{[0,T]}} \{v_t + (av)_x + Av\}\, udtdx +$$

$$+ \int_{G_{[0,T]}} vb\; dZ(t,\omega)dx + \int_{R^1} v(0,x)u_0(x)dx = 0,$$

where the second term on the right hand side means

$$\int_{G_{[0,T]}} vbdZ(t,\omega)dx = \int_{R^1} dx \int_0^T v(t,x)b(t,x)\, dZ(t,\omega)$$

and the integral $\int_0^T v(t,x)\, b(t,x)\, dZ(t,\omega)$ is the Wiener integral.

In the course of later discussions it will turn out to be clear that the definition is a natural extension of that of a weak solution of an equation like (4.3).

4. 3. The Existence and the Uniqueness of Solutions

In this section, we are concerned with the problems of existence and of uniqueness of solutions. Similary to those in initial value problems of linear equations, the problem of uniqueness in our case is also reduced to the same problem for linear homogeneous equations. However, as we have given the definition of solutions in a somewhat implicit form, it is not an obvious question to establish the uniqueness property.

We begin with the next

Lemma 1. If the functions $a(t,x)$, $A(t,x)$ and $f(t,x;\omega)$, which are defined on the slab $G_{[0,T]}$ and on $\Omega \times G_{[0,T]}$, satisfy the following conditions,

(c.1), $a(t,x)$ is continuous in t and is of C^2-class in x.

(c.2), $A(t,x)$ is continuous in t and is of C^1-class in x.

(c.3), $f(t,x;\omega)$ is almost surely a smmoth function in (t,x) and has a compact support in $G_{[0,T]}$.

Then, the next final value problem

$$(4.4) \qquad \frac{\partial}{\partial t} v(t,x;\omega) + \frac{\partial}{\partial x}\left\{ a(t,x)v(t,x;\omega)\right\} + A(t,x)v(t,x;\omega)$$

$$= f(t,x;\omega),$$

(4.5) $\quad v(T,x;\omega) = 0$,

almost surely has a solution $v(t,x;\omega)$ which has the same property of $f(t,x;\omega)$.

(Remark) Since the function $f(t,x;\omega)$ has the properties mentioned in (C.3), we can regard the equation (4.4) as a family of partial differential equations each of which is deterministic. Thus by the word "a solution of (4.4)", we convince it to mean such a random function that satisfies (4.4) in a usual sense for each fixed random parameter.

(Proof of Lemma 4. 1) For the proof, it suffices to see that the following function $v(t,x;\omega)$ has all the desired properties.

$$(4.6) \quad v(t,x;\omega) = \int_t^T f(s, X^{(t,x)}(s);\omega) \exp\{-\int_s^T A(r,x^{(t,x)}(r))dr\}ds,$$

where $A(t,x) = a_x(t,x) + A(t,x)$ and $X^{(t,x)}(s)$ $(0 \le t \le s \le T)$ is the solution of the following equation.

$$(4.7) \quad X^{(t,x)}(s) - x = \int_t^s a(r, X^{(t,x)}(r))dr,$$

which has a unique solution by virtue of the condition imposed on $a(t,x)$.

With this preparation, we can establish the next

<u>Theorem 4. 1.</u> (Uniqueness), If $a(t,x)$ and $A(t,x)$ are the functions described in Lemma 1, and if there exists a solution to the initial value problem (4.1), (4.2), then it is a unique L^2_{loc} $(\Omega\times G_{[0,T]})$-solution.

(Proof of Theorem 4. 1) Let $u_1(t,x;\omega)$ and $u_2(t,x;\omega)$ be solutions of the problem, and let us put $\tilde{u}(t,x;\omega) = u_1(t,x;\omega) - u_2(t,x;\omega)$.

Then $\tilde{u}(t,x;\omega)$ is a function that is continuous in t and x in the $L^2(\Omega)$-norm, and that satisfies the following equation with probability one for any smooth function $v(t,x)$ which has a compact support in $G_{[0,T]}$,

$$(4.8) \qquad \int_{G_{[0,T]}} v\{L(u_1) - L(u_2)\}\, dtdx = \int_{G_{[0,T]}} \{v_t + (av)_x + Av\}\, \tilde{u}\, dtdx = 0.$$

Let us define

$$(4.9) \qquad \tilde{u}_h^M (t,x;\omega) = \int_{G_{[0,T]}} \rho_h(t-s,\ x-y)\, \tilde{u}^M (s,y;\omega)\, dsdy$$

, where ρ_h is a two-dimensional molifier with radius h, and $\tilde{u}^M(t,x;\omega)$ is a function that is defined for a compact set $M(\subset G_{[0,T]})$ by the following

$$(4.10); \qquad \tilde{u}^M(t,x;\omega) = \tilde{u}(t,x;\omega), \quad \text{for } (t,x) \text{ in } M$$
$$= 0 \qquad \text{otherwise.}$$

Then $\tilde{u}^M_h(t,x;\omega)$ is a function which is almost surely smooth in (t,x) and has a compact support in $G_{[-h,T+h]}$. Following the previous lemma, we know that there exists a function $v^M_h(t,x;\omega)$ which is almost surely smooth in (t,x), has a compact support in $G_{[0,T]}$, and satisfies the relation $(v^M_h)_t + (av^M_h)_x + Av^M_h = \tilde{u}^M_h$ with probability one.

Now $v(t,x)$ in (4,8) being arbitrary, we have

$$(4.8)' \qquad \int_{G_{[0,T]}} \tilde{u}^M_h \tilde{u} \, dtdx = 0,$$

with probability one , by the substitution of $v^M_h(t,x;\omega)$ into $v(t,x)$ in (3.5), regarding that $v(t,x)$ is a realization of $v^M_h(t,x;\omega)$

On the other hand, it is not difficult to see from the definition of $\tilde{u}_h^M$ that the relation $\lim_{h \to 0} \tilde{u}_h^M (t,x;\omega) = \tilde{u}^M(t,x;\omega)$ holds with probability one for each fixed (t,x). From this fact, we can obtain, with the help of the Fatou's lemma, the following relation ;

$$0 = \lim_{h \to 0} E[\int_{G_{[0,T]}} (\tilde{u}_h^M) \tilde{u} \, dtdx] = E[\int_{G_{[0,t]}} (\tilde{u}^M) \tilde{u} \, dtdx]$$

$$= E[\int_M (\tilde{u})^2 \, dtdx] .$$

And this completes the proof.

In what follows, we are going to establish the existence theorem of solutions. The idea for this aim is to apply the characteristic method which is well known in the deterministic theory of partial differential equations. But in our case, we must introduce a certain technic because the equation contains the white noise which is not an oridinary random function.

In order to establish the existence theorem, we consider a family of stochastic processes which approximates the Brownian motion process $Z(t,\omega)$.

Given a family of partitions $\{ \Delta^{(n)};\ 0 \le t_1^{(n)} < t_2^{(n)} < \ldots < t_n^{(n)} \le T\}_{n=1}$ of the interval $[0, T]$ which has a property, $\lim_{n\to\infty} \max_{1\le i\le n-1} (t_{i+1}^{(n)} - t_i^{(n)}) = 0$, we construct a family of stochastic processes $\{Z^{(n)}(t,\omega),\ 0\leq t\leq T\}_{n=1}^{\infty}$ in the following manner;

$$Z^{(n)}(t,\omega) = Z(t_i^{(n)},\omega) + \Delta_i^{(n)} Z \frac{t - t_i^{(n)}}{{}_{i}^{(n)}}, \tag{4.11}$$

$$\text{for} \quad t \in [t_i^{(n)},\ t_{i+1}^{(n)}]$$

,where $\quad \Delta_i^{(n)} Z = Z(t_{i+1}^{(n)},\omega) - Z(t_i^{(n)},\omega), \quad \tau_i^{(n)} = t_{i+1}^{(n)} - t_i^{(n)}.$

As for this family of stochastic processes, it is easy to see that each element connverges in the mean to the Brownian motion process, as n tends to infinity and that the convergence is uniform in t. Moreover, we can see that each element of the family has such a nice property from the viewpoint of physics as the almost sure piecewise-smoothness.

Lemma 4. 2 For a piecewise continuous function $f(t)$ $(t \in [0,T])$, the following sequence of the Stiltjes integrals $\{S_n(f)\}_{n=1}^{\infty}$, the n-th term of which is defined by

(4.12) $$S_n(f) = \int_0^T f(t)\, dZ^{(n)}(t,\omega) = \sum_{i=1}^{n-1} \frac{\Delta_i^{(n)} Z}{\tau_i^{(n)}} \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} f(t)\, dt ,$$

converges in the mean to the Wiener integral $\int_0^T f(t)\, dZ(t,\omega)$:

(4.13) $$\lim_{n\to\infty} E\left\{ \int_0^T f(t)\, dZ(t,\omega) - \int_0^T f(t)\, dZ^{(n)}(t,\omega)\right\}^2 = 0.$$

(Remark) The result is a special case of the author's proposition stated in S. Ogawa [30], if the function is piecewise Hölder continuous with an exponent which is greater than $\frac{1}{2}$.

(Proof of Lemma 4. 2) Let us estimate the following quantity.

(4.14) $$\delta(\Delta^{(n)}) = \sum_{i=1}^{n-1} \left\{f(t_i^{(n)}) - \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} f(t) \frac{dt}{\tau_i^{(n)}} \right\}\Delta_i^{(n)} Z .$$

Since $E\{\Delta_i^{(n)} Z \cdot \Delta_j^{(n)} Z\} = 0$ for $i \neq j$, we have

$$E\left\{\delta^2(\Delta^{(n)})\right\} = E\left\{ \sum_{i=1}^{n-1} \frac{\Delta_i^{(n)} Z}{\tau_i^{(n)}} \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} (f(t_i^{(n)}) - f(t))\, dt\right\}^2$$

$$\leq \sum_{i=1}^{n-1} \frac{1}{\tau_i^{(n)}} \left(\int_{t_i^{(n)}}^{t_{i+1}^{(n)}} (f(t_i^{(n)}) - f(t))\, dt\right)^2$$

$$\leq \sum_{i=1}^{n-1} \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} (f(t_i^{(n)}) - f(t))^2 \, dt$$

Hence, we obtain $\lim_{n\to\infty} E\,\{\delta^2(\,\Delta^{(n)})\} = 0$, by virtue of the piecewise continuity of $f(t)$.

This completes the proof because the Wiener integral of $f(t)$ is defined in the following way,

$$\int_0^T f(t)\, dZ(t,\omega) = \lim_{n\to\infty} \sum_{i=1}^{n-1} f(s_i)\, \Delta_i^{(n)} Z \ ,$$

$$\text{with} \quad s_i^{(n)} \in [t_i^{(n)}\ ,\ t_{i+1}^{(n)}\ ]$$

, and is not affected by the choise of points $s_i^{(n)} (i=1, 2, \ldots, n-1)$,

Now let us consider the following initial value problem, corresponding to the original problem (3.1), (3.2).

$$(4.15) \qquad \frac{\partial}{\partial t} u^{(n)}(t,x;\omega) + a(t,x) \frac{\partial}{\partial x} u^{(n)}(t,x;\omega)$$

$$= A(t,x)\, u^{(n)}(t,x;\omega) + b(t,x) \frac{d}{dt} Z^{(n)}(t,\omega),$$

$$(4.16) \qquad u^{(n)}(0,x;\omega) = u_0(x)$$

Since the stochastic process $Z^{(n)}(t,\omega)$ which is defined in

(4.11), almost surely has a piecewise smooth sample function, the equation (4.15) almost surely has a meaning as a partial differential equation of the first order for each fixed random parameter ω. Thus we can apply the theory of partial differential equations to the present case, that is, we have the next

<u>Proposition 4. 1.</u> If the functions $a(t,x)$, $b(t,x)$, $A(t,x)$ and $u_0(x)$ satisfy the following conditions,

(C.4) $a(t,x)$ is continuous in t and lipschitz-continuous in x on $G_{[0,T]}$,

(C.5) $A(t,x)$ and $u_0(x)$ are continuous functions on $G_{[0,T]}$,

(C.6) $b(t,x)$ is continuous in t and piecewise continuous in x on $G_{[0,T]}$.

Then there exists a unique solution of the problem (4.15), (4.16) and the solution $u^{(n)}(t,x;\omega)$ also satisfies the following integral equation;

$$(4.17)\qquad u^{(n)}(t,x;\omega) - u_0(X^{(t,x)}(0)) = \int_0^t A(s, X^{(t,x)}(s))u^{(n)}(s,X^{(t,x)}(s);\omega)ds + \int_0^t b(s, X^{(t,x)}(s))\, dZ^{(n)}(s,\omega)$$

where $X^{(t,x)}(s)$ $(0 \le s \le t \le T)$ is a solution of the following

$$(4.18). \qquad x - X^{(t,x)}(s) = \int_s^t a(r, X^{(t,x)}(r))\, dr \; .$$

(Remark) (i) By the terminology "a solution of the problem", we intend to mean such a random function $u^{(n)}(t,x;\omega)$ that is almost surely continuous in (t,x) and that satisfies the following relation with probability one for an arbitrary smooth function $v(t,x)$ which has a compact support in $G_{[0,T]}$,

$$(4.19) \qquad \int_{G_{[0,T]}} v(t,x)\, L^{(n)}(u^{(n)})\, dtdx$$
$$= \int_{G_{[0,T]}} \left\{ v_t + (av)_x + Av \right\} u^{(n)}\, dtdx + \int_{G_{[0,T]}} vb\cdot dZ^{(n)}(t,\omega)\, dx + \int_{R^1} v(0,x)\, u_0(x)\, dx$$
$$= 0 \; .$$

(ii) In order to yield only the existence of solutions, we can replace condition (C.4) by a weaker one, (cf., E. D. Conway [4]).

The verification of the assertion is not difficult and is omitted here. However we must notice that the solution can be written in an explicit form as follows;

$$(4.20), \quad u^{(n)}(t,x;\omega) = u_0(X^{(t,x)}(0))\exp\{\int_0^t A(s, X^{(t,x)}(s))\,ds\}$$
$$+ \int_0^t b(s, X^{(t,x)}(s))\exp\{\int_s^t A(r, X^{(t,x)}(r))\,dr\}\,dZ^{(n)}(s,\omega).$$

As for this solution, we have the next

Lemma 4. 3. For any fixed (t,x) in $G_{[0,T]}$, $u^{(n)}(t,x;\omega)$ converges in the mean to a function $u(t,x;\omega)$, which is the unique solution of the following integral equation.

$$(4.21) \quad u(t,x;\omega) - u_0(X^{(t,x)}(0))$$
$$= \int_0^t A(s, X^{(t,x)}(s))\,u(s,x;\omega)\,ds$$
$$+ \int_0^t b(s, X^{(t,x)}(s))\,dZ(s,\omega),$$

where $X^{(t,x)}(s)$ $(0 \le s \le t \le T)$ is the function mentioned in (3.15).

(Proof of Lemma 4. 3) Suppose that there exists a unique solution $u(t,x;\omega)$ of (4.21). Let us put $\tilde{u}(t,r,x;\omega) = u(t, X^{(r,x)}(t);\omega)$ for any fixed $r(\ge t)$. Then it is easy to see that the function $\tilde{u}(t,r,x;\omega)$ must be a solution of the following equation, which is derived from (4.21) by the substitution of $X^{(r,x)}(t)$ into x.

$$(4.21)' \qquad \tilde{u}(t,r,x;\omega) - u_0(X^{(r,x)}(0)) = \int_0^t A(s, X^{(r,x)}(s))\, \tilde{u}(s,r,x;\omega)\, ds + \int_0^t b(s, X^{(r,x)}(s))\, dZ(s,\omega) .$$

On the other hand, it is easy to see that if equation (4.21)' has a solution $\tilde{u}(t,r,x;\omega)$ which is continuous in $r(\geq t)$ in $L^2(\Omega)$-sense, then the function $u(t,t,x;\omega)$ satisfies the equation (4.21). It is also not difficult to see that a solution of (4.21), if it exists, is unique in the sense of $L^2(\Omega)$ for each fixed t and x. Therefore the question of the existence of a unique solution of (4.21) is reduced to the question of the existence of a solution $u(t,r,x;\omega)$ of equation (4.21)' which is continuous in $r(\geq t)$ in the sense of $L^2(\Omega)$.

Applying the differential formula of Ito (K. Ito [20]), we have,

$$d_t[\, \tilde{u}(t,r,x;\omega) \exp \{- \int_0^t A(s, X^{(r,x)}(s))\, ds \}\,]$$
$$= \exp \{- \int_0^t A(s, X^{(r,x)}(s))ds\}\, b(t, X^{(r,x)}(t))\, dZ(t,\omega).$$

Thus we have,

$$\tilde{u}(t,r,x;\omega) = u_0(X^{(r,x)}(0)) \exp\{ \int_0^t A(s, X^{(r,x)}(s))\, ds \}$$
$$+ \int_0^t b(s, X^{(r,z)}(s)) \exp\{ \int_s^t A(p, X^{(r,x)}(p))\, dp \}\, dZ(s,\omega)$$

Since the function above is continuous in $r(\geq t)$ in the sense of $L^2(\Omega)$, we know that equation (4.21) has a unique solution which is given in an explicit form as follows

$$(4.22) \qquad u(t,x;\omega) = \underset{r \to t}{\text{l.i.m}}\, \tilde{u}(t,r,x;\omega)$$
$$= u_0(X^{(t,x)}(0)) \exp\{ \int_0^t A(s, X^{(t,x)}(s))\, ds\}$$
$$+ \int_0^t b(s, X^{(t,x)}(s)) \exp\{ \int_s^t A(p, X^{(t,x)}(p))\, dp \}\, dZ(s,\omega) \ldots\ldots$$

Applying the result of Lemma 4.2 to these functions (4.20), (4.22), we complete the proof of the proposition.

(Remark) Throughout the proof, we have used the fact that the Wiener integral $\int_0^t b(s, X^{(r,x)}(s))\, dZ(s,\omega)$ is continuous in r in $L^2(\Omega)$-sense for each fixed t and x. The verification of this fact is not difficult and is omitted here.

We are now in the position to state

Theorem 2. (Existence) Under the conditions (C.4) - (C.6), the initial value problem (4.1), (4.2) has a solution $u(t,x;\omega)$ and the solution satisfies the stochastic integral equation (4.21).

(Proof of Theorem 4. 2.) It may suffice to prove that the solution of (4.21) satisfies the condition S.2). For the solution of (4.21), we put

$$\int_{G_{[0,T]}} vL(n)\, dtdx = \int_{G_{[0,T]}} \{v_t + (av)_x + Av\}\, u\, dtdx + \int_{G_{[0,T]}} bv\, dZ(t,\omega)\, dx + \int_{t=0} v(0,x)\, u_0(x)\, dx \quad .$$

Then we have from (4.19), the following relation.

$$E\{ \int_{G_{[0,T]}} vL(u)\, dtdx \}^2 = E\{ \int_{G_{[0,T]}} \tilde{v}(u-u^{(n)})\, dtdx + \int_{R^1} dx \int_0^T (bv\, dZ - bv\, dZ^{(n)}_{\circ})\}^2$$

where $\tilde{v} = v_t + (av)_x + Av$.

Applying the Schwarz's inequality,

$$\leq 2\mu(E_1)\, [\int_{E_1} \pi_v(x)\, dx \int_0^T \tilde{v}^2\, E\{(u-u^{(n)})^2\}\, dt + \int_{E_1} dx\, E\{ \int_0^T vb\, dZ - \int_0^T vb\, dZ^{(n)}\}^2 \;]$$

where E_1 is a set such that $E_1 = \{x;\ v(t,x) \neq 0$, for some t in $[0,T]\}$, $\mu(E_1)$ is its Lebesgue measure and $\pi_v(x)$ is a measure of the set $\{t;\ v(t,x) \neq 0\}$ for a fixed x in R^1.

$$E\left\{\int_0^T vb\ dZ - \int_0^T vb\ dZ^{(n)}\right\}^2$$

$$\leq 2\left\{\int_0^T (vb)^2 dt + E\left(\sum_{i=1}^{n-1} \frac{\Delta_i^{(n)} Z}{\tau_i^{(n)}} \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} vb\ dt\right)^2\right\}$$

$$\leq 2\left\{\int_0^T (vb)^2 dt + \sum_{i=1}^{n-1} \frac{1}{\tau_i^{(n)}} \left(\int_{t_i^{(n)}}^{t_{i+1}^{(n)}} (vb)^2 dt\right)^2\right\}$$

$$\leq 4 \int_0^T (vb)^2 dt < \infty$$

Therefore we have,

$$\lim_{n\to\infty} \int_{E_1} dx\ E\left\{\int_0^T vb\ dZ(t,\omega) - \int_0^T vb\ dZ^{(n)}(t,\omega)\right\}^2 = 0$$

by virtues of Lemma 4.2. and the theorem of bounded convergence. As for the remaining term, it is easy to see with the help of Lemma 4.3, that it also tends to zero as n tends to infinity.

Hence we have obtained,

$$E\{\int_{G_{[0,T]}} vL(u)\,dt\,dx\}^2 = 0, \text{ for an arbitrary test function}$$

$v(t,x)$, and this completes the proof.

It must be remarked that the same assertions of Theorem 1 and 2, also hold for the initial value problem of equation (4.1) with such an initial data $u_0(x;\omega)$ that is continuous in the sense of $L^2(\Omega)$, since considerations stated in this section do not be affected by this modification.

Therefore we have the following assertion as a corollary of Theorem 4.1 and Theorem 4.2.

Corollary Let $a(t,x)$, $b(t,x)$, $A(t,x)$ and $u_0(x;\omega)$ be such functions that satisfy the following conditions;

(C.1)' $a(t,x)$ is continuous in t and of C^2-class in x with such a derivative $a(t,x)_x$ which is bounded on the slab $G_{[0,T]}$,

(C.2)' $A(t,x)$ is continuous in t and is of C^1-class in x, in $G_{[0,T]}$,

(C.3)' $b(t,x)$ is continuous in t and piecewise continuous in x,

(C.4)' $u_0(x;\omega)$ is a stochastic process which is in the class $L^2_{loc}(\Omega\times R^1)$, moreover is continuous in the sense of $L^2(\Omega)$.

Then the initial value problem of equation (4.1) with

$$(4.23) \qquad u(0,x;\omega) = u_0(x;\omega)$$

has a unique solution, which is expressed in an explicit form as follows;

$$(4.24) \qquad u(t,x;\omega) = u_0(X^{(t,x)}(0);\omega) \exp\{\int_0^t A(s, X^{(t,x)}(s))\, ds\} + \int_0^t b(s, X^{(t,x)}(s)) \exp\{\int_s^t A(p, X^{(t,x)}(p))\, dp\} dZ(s,\omega).$$

4. 4. Statistical Properties of the Additive Noise

From now on, we are interested in statistical effects of a presence of the white noise in such linear systems that are characterized by equations like (4.1). In a practical sense, the problem which has been treated in this paper may be interpreted as follows;

> "Assume that a signal u_0 was transmitted at time zero through a time-varying channel in which the signal travels with a velocity $a(t,x)$, with an atte-nuation rate $A(t,x)$ and is disturbed by the white noise that enters into the system as an external disturbance. Then in what kind of form does it be observed when it reaches to a receiver ?"

In the previous section we have proved that the initial value problem (4.1), (4.2) has a unique solution and the solution can be written in an explicit form;

$$u(t,x;\omega) = u_0(X^{(t,x)}(0)) \exp\{\int_0^t A(s, X^{(t,x)}(s))\, ds\} + \int_0^t b(s, X^{(t,x)}(s)) \exp\{\int_s^t A(r, X^{(t,x)}(r))\, dr\} dZ(s,\omega).$$

Also in a practical sense, it may seem to be interpreted as follows,

(i) The first term of the solution represents a signal form which the receiver might expect to observe when there are no disturbances in the transmission line. (In fact, it is a solution of the same problem of the equation with $b(t,x) = 0$.)

(ii) The second term represents, therefore, a noise that is observed as a result of the presence of the white noise in the transmission line, and may correspond to a quantity which is called "the additive noise" in the statistical theory of communications.

However such interpretations are not faithful to the practical situation, since we have $u(0,x;\omega) = u_0(x)$, which means that there are not any noises in the transmission line at time zero, while in a practical system, we observe a signal and a noise at any time and any place. Speaking rigorously, the problem that we have studied corresponds to a situation where the system, together with an external noise, were at an absolutely calm state when the original signal was set to travel at time zero and all the things begin to move as soon as the signal begins to travel.

As stated in 4. 1, we wish to treat the case where there exists, and existed through whole of the past, an external noise in the system. Following the condiderations explained above, it may turn out to be clear that the correct form (in a practical sense) of the observed signal is given as a solution of the initial value problem of equation (4.1) with the following initial data,

$$(4.25) \qquad u(0,x;\omega) = u_0(x;\omega) = u_0(x) + N(0,x;\omega)$$

where $N(0,x;\omega)$ is a noise observed at time zero and at x in R^1.

By the quantity $N(0,x;\omega)$ we wish to represent a noise which is in the system as a cosequence of the constant existence of the white noise, and we also wish to determine it as a special value of a stochastic process $\{N(t,x;\omega),\ x\in R^1\ -\infty<t<\infty\}$, each of specified values of which repersents a noise observed at (t,x).

Hence we are led to the next

Definition. We shall call a stochastic process $N(t,x;\omega)$ $(-\infty<t<\infty,\ x\in R^1)$ an "additive noise of the linear system L", provided

N.1), $N(t,x;\omega)$ is continuous in x in the sense of $L^2(\Omega)$, for each fixed t.

N.2), $\sup_{(t,x)\in R^2} E\{(N(t,x;\omega))^2\} < \infty$.

N.3), The next relation holds with probability one for each t, s and x,

$$(4.26) \qquad N(t+s,\ x;\omega) = L(s)\, N(t,x;\omega)\ ,$$

where $L(s)$ $(s\geq 0)$ is a mapping defined by

$$(4.27) \qquad L(s)\, N(t,x;\omega) = N(t, X^{(t+s,x)}(t);\omega) \exp\{\int_t^{t+s} A(p, X^{(t+s,x)}(p))\, dp\}$$

$$+ \int_t^{t+s} b(p, X^{(t+s,x)}(p)) \exp\{\int_p^{t+s} A(q, X^{t+s,x)}(q))\, dq\} dZ(p,\omega) .$$

It must be noticed that the mapping $L(s)$ $(s \geq 0)$ has such a property that $L(t)u_0(x) = u(t,x;\omega)$ which is the solution of our original initial value problem. We will call such a stochastic process $N(t,x;\omega)$ an "additive noise", because it satisfies the next

Proposition 4. 2. Let $\{N(t,x;\omega), -\infty<t<\infty, x \in R^1\}$ be an additive noise. Then under the assumptions (C.1)' - (C.4)' on coefficients $a(t,x)$, $b(t,x)$, $A(t,x)$ and $u_0(x)$, the initial value problem of equation (4.1) with the initial condition (4.25) has a unique solution $u(t,x;\omega)$, which can be written as follows;

$$u(t,x;\omega) = L(t)(u_0(x) + N(0,x;\omega)) = v_0(t,x) + N(t,x;\omega)$$

, where $v_0(t,x)$ is a solution of the following problem,

$$(4.28) \qquad (v_0)_t + a(v_0)_x = Av_0 , \quad \text{with} \quad v_0(0,x) = u_0(x) .$$

(Proof of Proposition) Since $N(t,x;\omega)$ is continuous in x, in $L^2(\Omega)$-sense, the assertion is a direct consequence of the previous corollary.

o

As a convention to study the additive noise, we extend the domain, on which $a(t,x)$, $b(t,x)$ and $A(t,x)$ were defined, to R^2 preserving the continuity properties that were imposed on them in Corollary.

Proposition 4. 3 If the functions $a(t,x)$, $b(t,x)$ and $A(t,x)$ satisfy the following conditions,

A.1) $A(t,x) < 0$ on R^2 ,

A.2) $$\sup_{(t,x)\in R^2} \int_{-\infty}^{t} b^2(s, X^{(t,x)}(s)) \exp\{2 \int_{s}^{t} A(p, X^{(t,x)}(p))\, dp\} ds < \infty,$$

in addition to those mentioned before.

Then there exists a unique additive noise and it is given in the next

$$N(t,x;\omega) = \int_{-\infty}^{t} b(s, X^{(t,x)}(s)) \exp\{ \int_{s}^{t} A(p, X^{(t,x)}(p))\, dp\} dZ(s,\omega). \tag{4.29}$$

(Proof of Proposition 4. 3) From (4.26) and (4.27), we have for any fixed $s(\le t)$,

$$N(t,x;\omega) = L(t-s)\, N(s,x;\omega)$$

$$= N(s, X^{(t,x)}(s);\omega) \exp\{ \int_{s}^{t} A(p, X^{(t,x)}(p))\, dp\}$$

$$+ \int_s^t b(p, X^{(t,x)}(p)) \exp\{ \int_p^t A(q, X^{(t,x)}(q))\, dq\} dZ(p,\omega).$$

Letting $s \to -\infty$, we have the expression (4.29) as a cosequence of assumptions (A.1), (A.2) and the condition(N.2).

On the other hand, it is easy to see that the stochastic process $N(t,x;\omega)$ given in (4.29) satisfies all the conditions (N.1) - (N.3). Q. E. D.

As for the statistical properties of the additive noise, we have the next proposition, which is a direct consequence of the theory of Wiener integrals (cf. K. Ito, [20]).

Proposition 4. 4 The additive noise $\{N(t,x;\omega), -\infty<t<\infty, x\in R^1\}$ which is derived above, is a Gaussian process in t for each fixed x, that has the following quantities.

(i) $E\{N(t,x;\omega)\} = 0$ for each (t,x).

(ii) $E\{N(t,x;\omega)\, N(s,x;\omega)\} = R(s,t;x)$

$$= \int_{-\infty}^{\min(s,t)} b(p, X^{(s,x)}(p))\, b(p, X^{(t,x)}(p)) \times \exp\{ \int_p^t A(q, X^{(t,x)}(q))\, dq + \int_p^s A(q, X^{(s,x)}(q))\, dq\} dp.$$

It may be of no use to investigate more about the statistical properties of $N(t,x;\omega)$, unless the practical situation where a phenomenon occurs is specified. So we will finish the discussion with the next

(Example). Let us consider the case where $a(t,x)$, $b(t,x)$ and $A(t,x)$ are such that,

(1) $a(t,x) = a_0 \quad (>0)$, (2) $A(t,x) = -A_0 \quad (<0)$,

(3) $b(t,x) = b_0$ for $x<D$ (a constant)

$= 0$ otherwise.

This example may correspond to a case where the signal propagates with a constant velocity a_0 through an area disturbed by the white noise. Since the functions described above satisfy the conditions (A.1), (A.2) in Proposition 4. 3. and assumptions in Corollary, we know that there exists a unique additive noise $N(t,x;\omega)$ following Proposition 4. 3..

Applying the formula (4.29) to this case, we have the expression of the additive noise as follows;

$$N(t,x;\omega) = \int_{-\infty}^{(D,x)} \exp(-A_0(t,-s))\, dZ(s,\omega),$$

where $(D,x) = \min\left(t,\ t+\dfrac{D-x}{a_0}\right)$.

For each specified x, this represents a velocity of the Ornstein-Uhlenbeck process. Hereafter we fix $x = \bar{x}$ (>D). The covariance function of this specified process is

$$(4.30)\qquad R(s,t) = \frac{b_0^2}{2A_0} \exp(2A_0(D-\bar{x})/a_0) \exp(-A_0|s-t|).$$

Therefore it is a stationry Gaussian process with the following power spectrum.

$$(4.31)\qquad S(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\tau) \exp(-i\lambda\tau)\, d\tau$$
$$= \frac{b_0^2}{\pi} \exp(2A_0(D-\bar{x})/a_0) \frac{1}{A_0^2 + \lambda^2},$$

where $R(\tau) = R(s,t)(|s-t|=\tau)$.

Since the covariance function $R(\tau)$ tends to zero as τ tends to infinity, the additive noise at $x=\bar{x}$, $N(t,\bar{x};\omega)$, has an ergodicity of the mixed type (cf. K. Ito [20]). Therefore it is possible in a practical situation to estimate the variables D, A_0, a_0, and b_0 by making the function $R(\tau)$ in the following manner,

$$R(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} N(t,\bar{x};\omega)\, N(t+\tau,\bar{x};\omega)\, dt ,$$

using the observation datas of the noise.

4. 5. Concluding Remarks

Starting from discussions on the initial value problem of a linear stochastic partial differential equation with the white noise as an external force term, we have reached to a precise notion of "the additive noise of a linear system". As we have explained in this paper, it means a noise which is inherent in the linear system considered. Following the results in this paper, we have something to refer to the statistical theory of communication. The first thing is about the hypothesis of the additive noise. Considerations in 4.3 and 4.4, make us to say that in such a linear system, the hypothesis must be slightly changed as follows; "transmitted signal u_0 is observed at (t,x) in the form, $L_0(u_0) + N(t,x;\omega)$", where $L_0(u_0)$ is a transformed signal, that one might expect to receive when one assumes that there are no external random disturbance in the transmission medium. However, as we have seen in Proposition 3, to know the transformation mechanics $u_0 \to L_0(u_0)$ exactly is to have the complete knowledges on the statistical properties of the additive noise. Therefore it seems to be somewhat impractical to employ the hypothesis. But the example in 44, encourages us to say that there is a situation where it is possible to estimate the transformation law $L_0(\cdot)$ by a practical observation of the additive noise. The author thinks that at this point the "analysis of noises" has its raison d'être.

The second thing is that the hypothesis of the additive noise may be valid only for such linear systems that possess the randomness as a result of the existence of an external random disturbance.

For example, it may fail to be valid in such a linear system where there is a fluctuation in a physical quantity which is inherent in the medium, (cf. S. Ogawa [31]). In such cases, the signal carries the noise whose statistical characters depend closely on the wave form of the pure signal transmitted. This may suggest us the necessity of an information theory where the noises are supposed to be dependent on the signals.

When the author refered to practical problems or situations in this paper, he considered of those in the statistical theory of communications, especially the problems in the "satelite communications". However, the analysis developed here, may be applicable to other random problems. For example, the analysis of a network of electrical circuits (e.g., the electrical circuit that is composed as a model of the "nerve axon"), in which each unit includes a resistence that generates a constant noise due to the thermodynamical reasons, may offer us an interesting problem related to the analysis in this paper.

CHAPTER V Wave Propagation in Random Media, (II)

5. 1. Introduction

Partial differential equations with random coefficients and the related problems were proposed by many mathematicians and physicists in connection with the analyses of random phenomena. Among these, the random transpostation equations were studied extensively by many authors, such as J. B. Keller [24], U. Frisch [10], and so on, in order to establish powerful tools for analysis of wave propagation in random media and other related physical problems (see, for example, U. Frisch [10]).

In this article, we introduce the following equation as a simple ultimate case of these random transportation equations,

$$(5.1) \qquad \frac{\partial}{\partial t} u + \{a(t,x) + \frac{d}{dt} Z_t(\omega)\} \frac{\partial}{\partial x} u = A(t,x)u + B(t,x),$$

and discuss the Cauchy problem for this equation with the initial data

$$(5.2) \qquad u(0,x;\omega) = u_0(x) \quad ,$$

where in the equation (5.1), $\{Z_t(\omega),\ t\geq 0\}$ is the R^1-valued Brownian motion process and $\frac{d}{dt} Z_t(\omega)$ is its formal derivative, namely the so called "White noise".

The author thought of the equation (5.1) as a limit case of the following random equation

$$(5.3) \qquad \frac{\partial u}{\partial t} + n(t,x;\omega) \frac{\partial u}{\partial x} = A(t,x)u + B(t,x),$$

where the random process $n(t,x;\omega)$ is taken to be somewhat regular, so that for each fixed random parameter ω, the equation has a definite meaning as a partial differential equation of the first order. It is well known that an equation of type (5.3) represents a phenomenon of wave propagation in a randomly fluctuated medium, with $n^{-1}(t,x;\omega)$ its index of reflaction. However, since the white noise is a physically unreasonable notion, the author does not know whether the same interpretation for the formal equation (5.1) is appropriate or not. The author's motivation for such a problem is not in showing a mathematical treatment for a problem of the wave propagation in random media. The author is motivated by the following primitive question; "Is it possible to consider a partial differential equation which has a diffusion process as its characteristics ?", or in other words; "Is it possible to construct the solution of a Cauchy problem of a linear parabolic equation by <u>the characteristics method</u> ?" In section 5.2, we will give the definition of the solution and show the existence theorem, which is the main theorem in this chapter. Moreover in section 5.3, we will derive the equation which the averaged quantity of our solution, constructed in the proof of the theorem, must satisfy. We will find in 5.2 that our definition of the solution is given as a limit formula of the usual definition of the weak solution for equations of type (5.3). Unfortunately, the uniqueness problem remains unsolved. However, the results obtained in section 5.2 ensures us to say that the equation (5.1) has a diffusion process as its characteristic line in a sense and it is possible to construct a solution of the Cauchy problem of a linear parabolic (heat) equation by the characteristic method. These facts may suggest to us that a linear parabolic equation and a linear partial differential equation of the first order are not essentially different from each other in a stochastic sense. The situation will be more clarified when we rewrite the formal random equation (5.1) into a stochastic partial differential equation using a B-derivative of the random function $u(t,x;\omega)$. But we must notice that this is no more than a conjecture, because we have not established the uniqueness of solutions. The details about this fact will be explained in the final section, 5. 4.

5. 2. Definition of the solution and the Existence Theorem

Definition We will call the random function $u(t,x;\omega)$ which is defined on the slab $G_{[0,T]} = \{(t,x);\ t\in[0,T],\ x\in R^1\}$ the solution of the Cauchy problem for the equation (5.1) with initial data (5.2), if $u(t,x;\omega)$ satisfies the following conditions;

u.1) For every fixed (t,ω) $(\in [0,T] \times \Omega)$, $u(t,x;\omega)$ is a Borel measurable and locally bounded function of x.

u.2) For any fixed x, $u(t,x;\omega)$ belongs to the class $\mathcal{S}^+$.

u.3) For any continuously differentiable function $\eta(t,x)$ with compact support (supp $\eta \subset G_{[0,T]}$), the next relation holds with probability one;

$$(5.4)\qquad \int_{R^1} dx \int_0^T \{\eta_t + (\eta a)_x\}\, u(t,x;\omega)dt + \int_{R^1} dx \int_0^T \eta_x u(t,x;\omega)dZ_t(\omega) + \int_{R^1} dx \int_0^T \{Au(t,x;\omega) + B\}\,\eta dt = 0 .$$

(Remark) Here the stochastic integral is taken in the sense of $\mathcal{J}^+_{\frac{1}{2}}(f)$. Hereafter we will denote the above equation (5.1), shortly by

$$\int_{R^1} dx \int_0^T L(u)dt = 0.$$

u.4) $u_0(0,x;\omega) = u_0(x)$ (with probability one).

We will see in the later discussions that the condition u.3) is a natural extension of the definition of a weak solution for the stochastic partial differential equation of type (5.3).

We are now in the position to state our

Theorem. If the functions $a(t,x)$, $A(t,x)$, $B(t,x)$ and $u_0(x)$ satisfy the following conditions;

A.1) $a(t,x)$, $A(t,x)$ and $B(t,x)$ are continuous on $G_{[0,T]}$.

A.2) $a(t,x)$, $A(t,x)$, $B(t,x)$ and $u_0(x)$ are twice continuously differentiable in x, and all the derivatives of these are bounded on $G_{[0,T]}$,

then the next system of stochastic integral equations has a unique solution with probability one and the solution $u(t,x;\omega)$ is also a solution for the Cauchy problem (5.1) ∿ (5.2),

$$(5.5) \qquad u(t,x;\omega) - u_0(\overset{(t,x)}{\xi}(0)) = \int_0^t \{A(\tau,\overset{(t,x)}{\xi}(\tau))\, u(\tau,\overset{(t,x)}{\xi}(\tau);\omega) + B(\tau,\overset{(t,x)}{\xi}(\tau))\}\, d\tau,$$

where $\overset{(t,x)}{\xi}(s)$ $(0 \le s \le t \le T)$ is a solution of the following,

$$(5.6) \qquad -\xi^{(t,x)}(s) + x = \int_s^t a(\tau,\xi^{(t,x)}(\tau))d\tau + Z(t,\omega) - Z(s,\omega).$$

The proof of this theorem is devided into two parts. In the first part, we will show that the system of equations (5.5), (5.6) has a unique solution and that the solution $u(t,x;\omega)$ satisfies conditions u.1), u.2), and u.4). In the second part we will show the remaining assertion of Theorem. Before going to do that, we must notice that we are solving (5.6) in s for a fixed t, (that is, we are solving (5.6) backward). Therefore the process $\{\xi^{(t,x)}(s), s\leq t\}$ determined by (5.6) is the "time reversed process" of the diffusion process $\{\tilde{\xi}^{(t,x)}(s), s\geq t\}$ which is determined by

$$(5.6)' \qquad \tilde{\xi}^{(t,x)}(s) - x = \int_t^s a(\tau,\tilde{\xi}^{(t,x)}(\tau))d\tau + Z_s(\omega) - Z_t(\omega).$$

The theorem states that the original problem (5.1), (5.2) is partly reduced to solving the integral equation (5.5) via the solution of (5.6). The readers should notice about the close analogy between our theorem and the usual theory of partial differential equation of first order. In this sense we will call the stochastic process $\{\xi^{(t,x)}(s), s\leq t\}$ the "characteristic line through (t,x)" of the equation (5.1), though the equivalence between the original Cauchy problem and the system of equations (5.5) and (5.6) has not yet been established.

(Proof of Theorem (1)) As for the existence and the uniqueness of solutions of (5.5) and (5.6), there will be no need to demonstrate the proof since it is a usual routine work using the method of successive approximation. In the course of doing so, it will turn out to be clear that the process $\{\overset{(t,x)}{\xi}(s),\ s\leq t\}$ is $\mathcal{N}^s_t \times \mathcal{B}_{[s,t]} \times \mathcal{B}_{R^1}$ - measurable in the triple (s,x,ω) for each fixed t, and therefore $u(t,x;\omega)$ is $\mathcal{N}^0_t \times \mathcal{B}_{[0,t]} \times \mathcal{B}_{R^1}$-measurable in (t,x,ω) since $u(t,x;\omega)$ is a Baire function of the aggregates $\{\overset{(t,x)}{\xi}(s),\ s\leq t\}$. We are now going to verify that $u(t,x;\omega)$ satisfies the condition (u.2). It is easy to see that the stochastic process $\overset{(t,x)}{\xi}(s)$ is uniformly $B^+(M_4)$-differentiable on $[s,t]$ for each fixed s and x, and that tha derivative $\zeta^t(s)$ is given as the solution of the following

$$(5.7), \qquad \zeta^t(s) + 1 = -\int_s^t a_x(\tau, \overset{(t,x)}{\xi}(\tau))\zeta^t(\tau)d\tau \quad .$$

Solving this, we have

$$(5.8) \qquad \zeta^t(s) = -\exp\left(-\int_s^t a_x(\tau, \overset{(t,x)}{\xi}(\tau))d\tau\right) \quad .$$

In fact we have from (5.6) and (5.7),

$$\overset{(t+h,x)}{\xi}(s) - \overset{(t,x)}{\xi}(s) - \zeta^t(s)(Z_{t+h} - Z_t) = -\int_s^t \{ a(\tau, \overset{(t+h,x)}{\xi}(\tau)) - a(\tau, \overset{(t,x)}{\xi}(\tau))$$

$$- a_x(\tau, \overset{(t,x)}{\xi}(\tau))\zeta^t(\tau)(Z_{t+h} - Z_t)\}\, d\tau$$

$$-\int_t^{t+h} a(\tau, \overset{(t+h,x)}{\xi}(\tau))d\tau \ , \ (h > 0).$$

Hence we get

$$\Delta_t(h,s)\xi = -\int_s^t a_x(\tau,\xi^{(t,x)}(\tau))\Delta_t(h,\tau)\xi d\tau$$
$$- \frac{1}{\sqrt{h}}\int_t^{t+h} a(\tau,\xi^{(t+h,x)}(\tau))d\tau$$
$$- \frac{1}{\sqrt{h}}\int_s^t (\xi^{(t+h,x)}(\tau) - \xi^{(t,x)}(\tau))\{a_x(\tau,\theta^h\xi^{(t,x)}(\tau))$$
$$- a_x(\tau,\xi^{(t,x)}(\tau))\}d\tau,$$

where

$$\Delta_t(h,s)\xi = \frac{1}{\sqrt{h}}\{\xi^{(t+h,x)}(s) - \xi^{(t,x)}(s) - \zeta^t(s)(Z_{t+h} - Z_t)\},$$

and

$$\theta^h\xi^{(t,x)}(\tau) = \xi^{(t,x)}(\tau) + \theta(\xi^{(t+h,x)}(\tau) - \xi^{(t,x)}(\tau)) \quad (0 \le \theta \le 1).$$

Taking the expectation of the forth power of both sides, we obtain

$$(5.9) \qquad E\{(\Delta_t(h,x)\xi)^4\} < K\{(\alpha^4T^3)\int_s^t E\{(\Delta_t(h,\tau)\xi)^4\}d\tau + F(h)\},$$

where K is a positive constant and $\alpha = \sup|a_x(t,x)|$ which is finite by virtue of the condition (A.2), and F(h) is such that,

$$F(h) = h \int_t^{t+h} E\{a(\tau, \xi^{(t+h,x)}(\tau))^4\} d\tau$$

$$+ T^3 \int_0^T \{E\{(\frac{\xi^{(t+h,x)}(\tau) - \xi^{(t,x)}(\tau)}{\sqrt{h}})^8\}\}^{\frac{1}{2}}$$

$$\times \{E\{a_x(\tau, \theta^h \xi(\tau)^{(t,x)}) - a_x(\tau, \xi^{(t,x)}(\tau))\}^8\}^{\frac{1}{2}} d\tau .$$

Since $a_x(t,x)$ is Lipschitz-continuous in x, and since $\xi^{(t,x)}(\tau)$ $(0 \le \tau \le t)$ is almost surely continuous in t, we conclude from above

$$(5.10) \qquad \lim_{h \to 0} F(h) = 0 \quad .$$

Now applying the Gronwall-Bellman lemma (see Codington-Levinson [5]) to the inequality (5.9), we obtain

$$(5.11) \qquad E\{(\Delta_t(h,s)\xi)^4\} \leqslant KF(h) \exp\{ - K\alpha^4 T^3 (t-s)\},$$

which yields with (5.10), $\lim_{h \to 0} E\{(\Delta_t(h,s)\xi)^4\} = 0$ for any fixed s $(\le t)$. Since the solution of (5.7) is almost surely continuous in t, we see from above, together with the previous Proposition 2.6, that $\xi^{(t,x)}(s)$ is uniformly $B^+(M_4)$-differentiable on $[s,T]$.

Let us put $\tilde{u}(r,t,x;\omega) = u(r, \xi^{(t,x)}(r);\omega)$ $(0 \leqslant r \leqslant t)$. Then we can rewrite (5.5) as

$$(5.5)' \qquad \tilde{u}(t,t,x;\omega) - u_0(\xi^{(t,x)}(0)) = \int_0^t \{A(\tau, \xi^{(t,x)}(\tau))\tilde{u}(\tau,t,x;\omega) + B(\tau, \xi^{(t,x)}(\tau))\} d\tau .$$

By virtue of the uniqueness property of the solution of (5.8), which implies the equality $\xi^{(t,\xi^{(s,x)}(t))}(q) = \xi^{(s,x)}(q)$ (almost surely for any $0 \le q \le t \le s$), we get the following

$$(5.12), \qquad \tilde{u}(t,s,x;\omega) - u_0(\xi^{(s,x)}(0)) = \int_0^t \{A(\tau,\xi^{(s,x)}(\tau))\tilde{u}(\tau,s,x;\omega) + B(\tau,\xi^{(s,x)}(\tau))\}d\tau ,$$

(almost surely for $s \ge t$).

Hence we have

$$(5.13). \qquad u(s,x;\omega) - u(t,x;\omega) = \int_t^s \{A(\tau,\xi^{(s,x)}(\tau))\tilde{u}(\tau,s,x;\omega) + B(\tau,\xi^{(s,x)}(\tau))\}d\tau + \tilde{u}(t,s,x;\omega) - \tilde{u}(t,t,x;\omega) .$$

The relation above shows that $u(t,x;\omega)$ is uniformly $B^+(M_4)$-differentiable for any fixed x, if $\tilde{u}(t,s,x;\omega)$ is uniformly differentiable in s for any x and that the derivative is given by

$$(5.14) \qquad \frac{\partial^+}{\partial^+ Z_t} u(t,x;\omega) = \frac{\partial^+}{\partial^+ Z_s} \tilde{u}(t,s,x;\omega) \Big|_{s=t} .$$

Following an analogous consideration in the previous case, we know that the derivative is given as the solution of the next

$$(5.15)\qquad \check{u}(t,s,x;\omega) = u_0'(\xi^{(s,x)}(0))\zeta^s(0) + \int_0^t A(\tau,\xi^{(s,x)}(\tau))\check{u}(\tau,s,x;\omega)\,d\tau + \int_0^t \{A_x(\tau,\xi^{(s,x)}(\tau))\tilde{u}(\tau,s,x;\omega) + B_x(\tau,\xi^{(s,x)}(\tau))\}\zeta^s(\tau)d\tau,$$

where

$$\check{u}(t,s,x;\omega) = \frac{\partial^+}{\partial^+ Z_t}\tilde{u}(t,s,x;\omega)$$

and

$$u_0'(\xi) = \frac{d}{d\xi}u_0(\xi) \quad .$$

Since the solution $\check{u}(t,s,x;\omega)$ of (5.15) is almost surely continuous in t, s, we know from (5.14) and Proposition 2.6 that $u(t,x;\omega)$ is uniformly $B^+(M_4)$-differentiable on the interval [0,T] for any fixed x.

Now we have checked the conditions u.1), u.2) and u.4) for the solution $u(t,x;\omega)$ of (5.5) except the localy boundedness in x. However it is obvious that $u(t,x;\omega)$ is bounded in x. We can see this by making an usual apriori estimate of $u(t,x;\omega)$.

o

In order to check the remaining condition u.3), we will prepare some lemmas.

We consider the Cauchy problem for the following random equation.

(5.16) $$\frac{\partial}{\partial t}u_n(t,x;\omega) + \alpha_n(t,x;\omega)\frac{\partial}{\partial x}u_n(t,x;\omega) = A(t,x)\, u_n(t,x;\omega) + B(t,x) ,$$

(5.17) $$u_n(0,x;\omega) = u_0(x) \qquad \text{a.s.} ,$$

where $\alpha_n(t,x;\omega) = a(t,x) + \frac{d}{dt} Z_t^{(n)}(\omega)$ and $\{Z_t^{(n)}(\omega),\ t\geq 0\}$ is the approximate process of Brownian motion which was defined in (4.11). Since the stochastic process $\frac{d}{dt} Z_t^{(n)}(\omega)$ almost surely has a piecewise continous sample path, the equation (5.15) has a definite meaning as a partial differential equation for each fixed parameter ω.

Now as for this problem we have the following

Lemma 5. 1. Under the same assumptions on $a(t,x)$, $A(t,x)$, $B(t,x)$ and $u_0(x)$,

(i) the following system of equations almost surely has a unique solution.

$$(5.18)\qquad u_n(t,x;\omega) - u_0(\xi_n^{(t,x)}(0)) = \int_0^t \{A(\tau,\xi_n^{(t,x)}(\tau))u_n(\tau,\xi_n^{(t,x)}(\tau);\omega) + B(\tau,\xi_n^{(t,x)}(\tau))\}d\tau \ ,$$

$$(5.19)\qquad -\xi_n^{(t,x)}(s) + x = \int_s^t a(\tau,\xi_n^{(t,x)}(\tau))d\tau + Z_t^{(n)}(\omega) - Z_s^{(n)}(\omega) \ ,$$

$(0 \leq s \leq t \leq T)$, and

(ii) the solution $u_n(t,x;\omega)$ is also a weak solution for the Cauchy problem (5.16) $\sim$ (5.17), namely the next relation holds with probability one.

$$(5.20)\qquad \int_{R^1} dx \int_0^T \{\eta_t + (\eta a)_x\}u_n dt + \int_{R^1} dx \int_0^T \eta_x u_n dZ_t^{(n)}(\omega) + \int_{R^1} dx \int_0^T \{Au_n + B\}\, dt = 0$$

for an arbitrary function $\eta(t,x)$ which is continuously differentiable and has a finite support.

(Proof of Lemma 5. 1.) Since the approximate process $Z_t^{(n)}(\omega)$ almost surely has a piecewise continuous sample function, it will be obvious from the assumptions A.1), A.2) that the equations (5.18), (5.19) almost surely has a unique solution. The remaining assertion (ii) will be verified using the theorm of E. D. Conway (Theorem 2 in E. D. Conway [4]). Q. E. D.

Following the discussions of E. D. Conway ([4]), we can write the explicit form of the solution $u_n(t,x;\omega)$ of (5.18);

$$(5.21) \qquad u_n(t,x;\omega) = u_0(\xi_n^{(t,x)}(0)) \exp\{ \int_0^t A(\tau,\xi_n^{(t,x)}(\tau))\, d\tau \} + \int_0^t B(s,\xi_n^{(t,x)}(s)) \exp\{ \int_s^t A(\tau,\xi_n^{(t,x)}(\tau))d\tau \}ds.$$

As for this solution we have the next

Lemma 5. 2.

(i) The function (5.21) converges in quadratic mean to the solution of (5.5) uniformly in t for each fixed x.

(ii) For each fixed x, the function $u(t,x;\omega)$ and the family $\{u_n(t,x;\omega)\}_{n=1}^{\infty}$ satisfy the conditions f.1) ∿ f.3) (in Chapter III).

(Remark) From the result (i), we know the explicit formula of the solution $u(t,x;\omega)$ of (5.5), that is;

$$(5.22) \qquad u(t,x;\omega) = u_0(\xi^{(t,x)}(0)) \exp\{ \int_0^t A(\tau,\xi^{(t,x)}(\tau))d\tau + \int_0^t B(\tau,\xi^{(t,x)}(\tau)) \exp\{ \int_\tau^t A(s,\xi^{(t,x)}(s))ds\}d\tau.$$

(Proof of Lemma 5. 2.) First we notice that for any fixed t ($\leq s$) and x, the solution $\xi_n^{(s,x)}(t)$ of (5.19) converges uniformly in s to the solution $\xi^{(s,x)}(t)$ of (5.6), in quadratic mean. Let us put

$$\tilde{u}(t,s,x;\omega) = u(t,\xi^{(s,x)}(t);\omega),$$

$$\tilde{u}_n(t,s,x;\omega) = u_n(t,\xi_n^{(s,x)}(t);\omega) \quad (s \geq t),$$

then we have

$$\tilde{u}_n(t,s,x;\omega) - u_0(\xi_n^{(s,x)}(0)) = \int_0^t \{A(\tau,\xi_n^{(s,x)}(\tau))\tilde{u}_n(\tau,s,x;\omega) + B(\tau,\xi_n^{(s,x)}(\tau))\}d\tau \quad . \tag{5.23}$$

For an arbitrarily fixed s ($\geq t$) we get from (5.12), (5.23) and the assumptions A.1), A.2) the following inequality;

$$\begin{aligned} |\tilde{u}(t,s,x;\omega) - \tilde{u}_n(t,s,x;\omega)| &< U_1 |\xi^{(s,x)}(0) - \xi_n^{(s,x)}(0)| \\ &+ (A_1U_0 + M_0)\int_0^t |\xi^{(s,x)}(\tau) - \xi_n^{(s,x)}(\tau)|d\tau \\ &+ M_0\int_0^t |\tilde{u}(\tau,s,x;\omega) - \tilde{u}_n(\tau,s,x;\omega)|d\tau \end{aligned}$$

where $A_1 = \sup|A_x|$, $U_1 = \sup|u_0'|$, $M_0 = \max(\sup|A|, \sup|B|)$, and U_0 is a constant such that $U_0 \geq \sup|u_n|$ for all $n \geq 0$, which are all finite.

Hence,

$$E\{|\tilde{u}(t,s,x;\omega) - \tilde{u}_n(t,s,x;\omega)|^2\} \leq 3U_1^2 E\{|\xi^{(s,x)}(0) - \xi_n^{(s,x)}(0)|^2\}$$

$$+ 3T(A_1U_0 + M_0)^2 \int_0^t E\{|\xi^{(s,x)}(\tau) - \xi_n^{(s,x)}(\tau)|^2\}d\tau$$

$$+ 3M_0^2 T \int_0^t E\{|\tilde{u}(\tau,s,x;\omega) - \tilde{u}_n(\tau,s,x;\omega)|^2\}d\tau \quad .$$

Applying the Gronwall-Belman's lemma, we obtain

$$E\{|\tilde{u}(t,s,x;\omega) - \tilde{u}_n(t,s,x;\omega)|^2\}$$

$$\leq 3[U_1^2 E\{|\xi^{(s,x)}(0) - \xi_n^{(s,x)}(0)|^2\} + T(A_1U_0 + M_0)^2$$

$$\times \int_0^T E\{|\xi^{(s,x)}(\tau) - \xi_n^{(s,x)}(\tau)|^2\}d\tau\ ]\ \exp\ (3M_0^2Tt) \ .$$

Thus we have

$$\lim_{n\to\infty} \sup_{t\in[0,T]} E\{|u(t,x;\omega) - u_n(t,x;\omega)|^2\}$$

$$= \lim_{n\to\infty} \sup_{t\in[0,T]} E\{|\tilde{u}(t,t,x;\omega) - \tilde{u}_n(t,t,x;\omega)|^2\} = 0$$

for any fixed x.

(ii) It is sufficient to check the condition f.3).
Let $\check{u}_n(t,s,x;\omega)$ $(s\geq t)$ be the solution of the following

$$(5.24)\qquad u_n(t,s,x;\omega) = u_0'(\overset{(s,x)}{\xi_n(0)})\zeta_n^s(0) + \int_0^t A(\tau,\overset{(s,x)}{\xi_n(\tau)})\check{u}_n(\tau,s,x;\omega)d\tau$$
$$+ \int_0^t \{A_x(\tau,\overset{(s,x)}{\xi_n(\tau)})\tilde{u}_n(\tau,s,x;\omega) + B(\tau,\overset{(s,x)}{\xi_n(\tau)})\}\zeta_n^s(\tau)d\tau,$$

where $\zeta_n^s(r)$ is the solution of the following

$$(5.25),\qquad \zeta_n^s(r) + 1 = -\int_r^s a_x(\tau,\overset{(s,x)}{\xi_n(\tau)})\zeta_n^s(\tau)d\tau \quad .$$

After a formidable computation we find that the sequence of random functions $\{u_n(t,t,x;\omega)\}_{n=1}^{\infty}$ posseses the properties $\alpha.1) \sim \alpha.3)$. Therefore the proof will be completed when we show the next

$$(5.26),\qquad \lim_{n\to\infty} \max_{p\in\Delta^{(n)}} E\{|\check{u}(p,p,x;\omega) - \check{u}_n(p,p,x;\omega)|^2\} = 0 \quad .$$

Since it is only a routine work to check (5.26) and is omitted here. The reader can get the conclusion using the Gronwall-Bellman's lemma, the boundedness of $\zeta_n^s(t)$, $\zeta^s(t)$ and the following fact;

$$\lim_{n\to\infty} \sup_{s\geq t} \{|\zeta_n^s(t) - \zeta^s(t)|^2\} = 0,$$

for any fixed t $(s\geq t)$. Q. E. D.

Lemma 5.3 For any compact set G in R^1, the next holds.

$$\lim_{n\to\infty}\int_G E\{\int_0^T u(t,x;\omega)dZ_t(\omega) - \int_0^T u_n(t,x;\omega)dZ_t^{(n)}(\omega)\}^2 dx = 0 ,$$

where $u(t,x;\omega)$ and $u_n(t,x;\omega)$ are the functions just mentioned in Lemma 5.2.

(Proof of Lemma 5.3) We notice that the convergence of the following

$$\lim_{n\to\infty}\max_{p\in\Delta^{(n)}}\sup_{p'\in[p,p(n)]} E\{\frac{1}{\sqrt{\tau_p^{(n)}}}\{u_n(p',x;\omega) - u_n(p,x;\omega) - \check{u}_n(p,x;\omega)(Z_{p'}^{(n)} - Z_p^{(n)})\}\}^4 = 0 ,$$

is uniform in x, since the convergence of the following limit is also uniform in x for any fixed s ($\lesssim$ p),

$$\lim_{n\to\infty}\max_{p\in\Delta^{(n)}}\sup_{p'\in[p,p(n)]} E\{\frac{1}{\sqrt{\tau_p^{(n)}}}\{\xi_n^{(p',x)}(s) - \xi_n^{(p,x)}(s) - \zeta_n^p(s)(Z_{p'}^{(n)} - Z_p^{(n)})\}\}^4 = 0 .$$

Therefore for an arbitrary positive number ε, there exists a positive integer n_0 such that for any integer $m(\geq n_0)$, the next holds for any pair (p,p') $(p\in\Delta^{(n)}, p'\in[p, p(m)])$,

$$(5.27)\quad E\{\frac{1}{\sqrt{\tau_{p'}}}\{u_m(p',x;\omega) - u_m(p,x;\omega) - \check{u}_m(p,x;\omega)(Z_{p'}^{(m)} - Z_p^{(m)})\}\}^4 < \varepsilon$$

Let us make an estimate of the quantity $N(u_n) = E\{\int_0^T u_n(t,x;\omega)dZ_t^{(n)}\}^2$.

We have

$$(5.28)\quad N(u_n) = E\{\sum_{i=0}^{n-1} \frac{\Delta_i^{(n)}Z}{\tau_i^{(n)}} \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} u_n(t,x;\omega)\, dt\}^2$$

$$= E\{\sum_{i=0}^{n-1} (\frac{\Delta_i^{(n)}Z}{\tau_i^{(n)}})^2 (\int_{t_i^{(n)}}^{t_{i+1}^{(n)}} u_n(t,x;\omega)\, dt)^2\}$$

$$+ 2E\{\sum_{i>j}^{n-1} (\frac{\Delta_j^{(n)}Z}{\tau_j^{(n)}})(\frac{\Delta_i^{(n)}Z}{\tau_i^{(n)}})(\int_{t_j^{(n)}}^{t_{j+1}^{(n)}} (u_n(t,x;\omega) - u_n(t_j^{(n)},x;\omega)dt)$$

$$\times (\int_{t_i^{(n)}}^{t_{i+1}^{(n)}} (u_n(t,x;\omega) - u_n(t_i^{(n)},x;\omega))dt)\}$$

$$\leq \sum_{i=0}^{n-1} d_n^2 \tau_i^{(n)} + 2\sum_{i>j}^{n-1} \frac{1}{\sqrt{\tau_i^{(n)}}}\frac{1}{\sqrt{\tau_j^{(n)}}}$$

$$\times \{(\tau_i^{(n)})^3 \int_{t_i^{(n)}}^{t_{i+1}^{(n)}} E(u_n(t,x;\omega) - u_n(t_i^{(n)},x;\omega))^4 dt\}^{\frac{1}{4}}$$

$$\times \{(\tau_j^{(n)})^3 \int_{t_j^{(n)}}^{t_{j+1}^{(n)}} E(u_n(t,x;\omega) - u_n(t_j^{(n)},x;\omega))^4 dt\}^{\frac{1}{4}}$$

, where $d_n = \sup_{(t,x)\in G_{[0,T]}} |u_n(t,x;\omega)|$.

Taking sufficiently large n, which assures the inequality (5.27) for an arbitrarily fixed positive number ε , we have

$$E\,\{u_n(t,x;\omega) - u_n(t_i^{(n)},x;\omega)\}^4 \leq \{8\ \varepsilon(t - t_i^{(n)})^2 + 24\frac{(t - t_i^{(n)})^4}{(\tau_i^{(n)})^2}\}$$

$$\times\, E\{|\check{u}_n(t_i^{(n)},x;\omega)|^4\} \quad ,$$

for any $t_i^{(n)}$ in $\Delta^{(n)}$ and t in $[t_i^{(n)}, t_{i+1}^{(n)})$.

Thus we obtain from (5.28),

$$N(u_n) \leq d_n^2\, T + 2\sum_{i>j}^{n-1} \bar{d}_i^{(n)}\bar{d}_j^{(n)}\tau_i^{(n)}\,\tau_j^{(n)} \quad ,$$

$$\text{where}\quad \bar{d}_i^{(n)} = \{\frac{8}{3} + \frac{24}{5}\,E(|\check{u}_n(t_i^{(n)},x;\omega)|^4)\}^{\frac{1}{4}} \quad .$$

On the other hand we know from equations (5.18), (5.24) and the assumption A,2) that the families of random functions $\{u_n(t,x;\omega)\}_{n=1}^{\infty}$ and $\{\check{u}_n(t,x;\omega)\}_{n=1}^{\infty}$ are uniformly bounded on $G_{[0,T]}$ for any fixed ω . Hence from the inequality above we know that the quantity $N(u_n)$ is uniformly bounded in x. Since the same fact is true for the quantity $N(u) = E\{\int_0^T u(t,x;\omega)\, dZ(t,\omega)\}^2$, we get the conclusion by the theorem of bounded convergence and the result (ii) of Lemma 5.2.

Q. E. D.

Let us complete the proof of our theorem.

(Proof of Theorem, II)   By virtue of the previous Lemma 5.1, we have the next

$$(5.29). \qquad \int_{R^1} dx \int_0^T \eta L_n(u_n)dt = 0 \quad \text{(with probability one)},$$

where $\quad L_n(u_n) = \dfrac{\partial}{\partial t} u_n + \alpha_n \dfrac{\partial}{\partial x} u_n - Au_n - B \quad .$

Hence,

$$E \left\{ \left| \int_{R^1} dx \int_0^T L(u)dt \right|^2 \right\} \leqslant 3E\left\{ \left| \int_{R^1} dx \int_0^T (u - u_n)(\eta_t + (\eta a)_x)dt \right|^2 \right\}$$

$$+ 3E\left\{ \left| \int_{R^1} dx \int_0^T (udZ_t(\omega) - u_n dZ_t^{(n)}(\omega)) \right|^2 \right\}$$

$$+ 3E\left\{ \left| \int_{R^1} dx \int_0^T A(u - u_n)dt \right|^2 \right\}$$

$$\leqslant 3\mu(G)\left\{ \int_G \chi_n(x)dx \int_0^T E\{\tilde{A}^2(u - u_n)^2\}\, dt \right.$$

$$\left. + \int_G dx\, E\left\{ \int_0^T \eta u dZ_t(\omega) - \int_0^T \eta u_n dZ_t^{(n)}(\omega) \right\}^2 \right\}$$

where $\tilde{A} = \eta_t + (\eta a)_x + \eta A$, which is bounded on $G_{[0,T]}$ by virtue of the assumption A,2), and $G = \{ x; \eta(t,x) \neq 0$ for some t in $[o,T]\}$ $\mu(G)$ is its Lebesgue measure and $\chi_\eta(x)$ is the Lebesgue measure of the set $\{t; \eta(t,x) \neq 0\}$ for a fixed x in R^1. Since the result of Lemma 5.3 is not violated by replacing u and u_n by ηu and ηu_n respectively, we get from the estimate above and the preceeding Lemma 5.3 and 5.2

$$E\{ \left|\int_{R^1} dx \int_0^T \eta L(u)dt\right|^2 \} = 0 \quad .$$

This completes the proof of our theorem.

5. 3, The Equation for the Averaged Solution .

Let us put $\tilde{u}(t,x) = E\,\{u(t,x;\omega)\}$, where $u(t,x;\omega)$ is the solution just constructed in our theorem. The aim in this section is to derive the equation which the averaged quantity $\tilde{u}(t,x)$ must satisfy.

For the brevity of considerations, hereafter we suppose that $\tilde{u}(t,x)$ is of C^2-class in x for each t and $\tilde{u}_x(t,x)$, $\tilde{u}_{xx}(t,x)$ are bounded on $G_{[0,T]}$.

Then we have

$$\frac{1}{h}\left[\,\tilde{u}(t+h,x) - \tilde{u}(t,x)\,\right] = \frac{1}{h}E\{\,\tilde{u}(t+h,x) - \tilde{u}(t,\ \xi^{(t+h,x)}(t)\ )\}$$

$$+\frac{1}{h}\,E\{\tilde{u}(t,\ \xi^{(t+h,x)}(t)\ ) - \tilde{u}(t,\ \xi^{(t+h,x)}(t+h))\}$$

, for $0\leq t\leq t+h\leq T$.

It is easy to verify that the first term on the right hand side tends to $A(t,x)\tilde{u}(t,x) + B(t,x)$ as h tends to zero. As for the second term, we have by the Taylor expansion,

$$\frac{1}{h}\,E\{\tilde{u}(t,\ \xi^{(t+h,x)}(t)\ ) - \tilde{u}(t,\ \xi^{(t+h,x)}(t+h)\ )\}$$

$$= \frac{1}{h}\,E\{\tilde{u}_x(t,x)(\ \xi^{(t+h,x)}(t) - \xi^{(t+h,x)}(t+h)\ )\}$$

$$+\frac{1}{h}\,E\{\ \frac{1}{2}\,\tilde{u}_{xx}(t,\ \Theta_h\xi)(\ \xi^{(t+h,x)}(t) - \xi^{(t+h,x)}(t+h)\ )^2\}$$

where $\Theta_h \xi = x + \theta(\xi^{(t+h,x)}(t) - x)$ $(0 \leq \theta \leq 1)$.

Since $\tilde{u}_x(t,x)$ and $\tilde{u}_{xx}(t,x)$ are bounded, we get by taking the limit $h \to 0$,

$$\lim_{h \to 0} \frac{1}{h} E\{ \tilde{u}(t, \xi^{(t+h,x)}(t)) - \tilde{u}(t, \xi^{(t+h,x)}(t+h))\} = - a(t,x)\tilde{u}_x + \frac{1}{2}\tilde{u}_{xx} .$$

Thus we have reached the following

Proposition . Under the prescribed assumptions, $\tilde{u}(t,x)$ becomes a solution of the following Cauchy problem ;

$$\frac{\partial}{\partial t} \tilde{u} + a(t,x) \frac{\partial}{\partial x} \tilde{u} = \frac{1}{2} \frac{\partial^2}{\partial x^2} \tilde{u} + A(t,x)\tilde{u} + B(t,x) , \tag{5.30}$$

$$\tilde{u}(0,x) = u_0(x) . \tag{5.31}$$

From this result we know that what we have done is an interpretation for the Cauchy problem of a linear parabolic equation from the viewpoint of that of a partial differential equation of the first order, via the notion of a stochastic characteristic . In the present case we used as a stochastic characteristic through a point (t,x), the stochastic process $\{ \xi^{(t,x)}(s) , s \leq t\}$ which is the time-reversed process of the diffusion process determined by (5.6)' . In this sense we may say that a parabolic equation like (5.30) has a diffusion process as its characteristic line.

5. 4, Concluding Remarks.

In this chapter we have treated the problem using the notion of B-derivateves and the new type of stochastic integral. The necessity of these tools is based on the definition of our problem which was given as an analogy, in a stochastic sense, to the usual definition of the weak solution of a partial differential equation. The author thinks that there may be another way of consideration for such a problem. However he thinks that our approach is a natural one as far as we consider the equation (5.1) to be an ultimate case of these equations of type (5.3). At the first moment the author wished to establish the equivalence between a linear parabolic equation of type (5.30) and a stochastic linear partial differential equation (5.1) which is an equation of the first order in its form. But it remains as a conjecture, since we do not know the uniqueness of the solution. However, we have found that it is possible to construct a solution of a linear parabolic equation by the characteristic method and that an equation of type (5.1) is the equation which has a diffusion process as its characteristic line.

We finish the discussion with a comment, which will give us a light to the question of understanding a linear parabolic equation in a sense of partial differential equations of the first order.

Let us consider the same problem for the following equation :

$$\frac{\partial}{\partial t} u + \frac{dZ}{dt} \frac{\partial}{\partial x} u = Au \quad . \tag{5.31}$$

Then, following our Theorem, we have the solution which can be written as follows ;

$$u(t,x;\omega) = u_0(\overset{(t,x)}{\xi}(0))\exp\{\int_0^t A(\tau, \overset{(t,x)}{\xi}(\tau))d\tau\}, \tag{5.32}$$

where $\overset{(t,x)}{\xi}(\tau) = x + Z(\tau,\omega) - Z(t,\omega) \ , \quad (\tau \leq t).$

Computing the B^+-derivative of this solution, we find that $u(t,x;\omega)$ satisfies the following

$$\frac{\partial^+}{\partial^+ Z_t} u(t,x;\omega) + \frac{\partial}{\partial x} u(t,x;\omega) = 0 \ . \tag{5.33}$$

This is a partial differential equation of the first order and has a definite meaning. The previous proposition tells us that the function $\tilde{u}(t,x) = E\{u(t,x;\omega)\}$ is a solution of the next;

$$\frac{\partial}{\partial t}\tilde{u} = \frac{1}{2}\frac{\partial^2}{\partial x^2}\tilde{u} + A(t,x)\tilde{u}, \quad \tilde{u}(0,x) = u_0(x) \ . \tag{5.34}$$

The equation (5.33) suggests to us that a linear parabolic equation can really be reduced to a partial differential equation of the first order by replacing the time scale dt by $dZ(t,\omega)$.

However it also states that the time scale $dZ(t,\omega)$ is too large to catch $u(t,x;\omega)$ precisely, while the scale dt is too small. Because we know that the equation (5.33) with initial data $u_0(x)$ does not have a unique solution. In fact it is easy to check that the function $u(t,x;\omega) = u_0(\xi^{(t,x)}(0))$ is also a solution for the same problem. It will be of interest in its self to investigate additional conditions which will yield the uniqueness of a solution of such stochastic partial differential equation. All the problems are postponed to the next occasion.

CHAPTER VI Uniqueness of Solutions of Brownian Particle Equations.

6. 1. Introduction

In chapter V, we have introduced a stochastic partial differential equation, that we want to call "a brownian particle equation" following [33] and [34], and we have considered the Cauchy problem of this type of equation ;

$$(5.1) \qquad \frac{\partial}{\partial t} u + [a(t,x) + \frac{d}{dt} Z_t(\omega)] \frac{\partial}{\partial x} u = A(t,x)u + B(t,x),$$

$$(5.2) \qquad u(0,x,\omega) = u_0(x).$$

We have established a theorem for existence of solutions by constructing one following the method of characteristics. But we have left the uniqueness problem unsolved, which is still difficult to give a rigorous answer because of the impliciteness of the definition of solutions.

However, one of the important purposes of our theory is to know a certain equivalence between equations of parabolic type and those of the first order, a special case of hyperbolic equations. For equations like (5.1), this aim is achieved by taking the expectation of the stochastic solution constructed in Theorem(Chap.V). In other words, it is not necessary to give a uniqueness theorem in a rigorous sens, but it may be sufficiant for the present aim to assure the uniqueness of the expectations of solutions. The essential ides in what follows is to introduce a class of solutions called "characteristic", which really characterizes the stochastic solutions of (5.1)-(5.2) whose expectations solve a parabolic equation. In other words, the problem of uniqueness in this sense is reduced to the same problem of solutions of corresponding parabolic equations.

In the following, we consider the Cauchy problem of a system of brownian particle equations ;

$$(6.1)\qquad \frac{\partial}{\partial t} u^i + [a^i(t,x) + d_i \frac{d}{dt} Z_t(\omega)] \frac{\partial}{\partial x} u^i = \sum_{j=1}^{N} A_{ij}(t,x) u^j + B^i(t,x),$$

$$(6.2)\qquad u^i(0,x,\omega) = u_0^i(x) \qquad (t,x,\omega) \in [0,T]\times R^1 \times \Omega$$

where $d_i (i = 1,2,..,N)$ are real constants.

<u>Definition</u> A random function $\underline{\underline{u}} = (u^1,...,u^N)$ is called a solution of the problem (6.1)-(6.2) provided that each element u^i of $\underline{\underline{u}}$ satisfies the conditions (u,1),(u,2) in Chapter V, and that

(u,3)' For any continuously differentiable function $\underline{\underline{v}} = (v^1,..,v^N)$ with compact support, such that $v^i(T,x) = 0$ $(i = 1,..N)$, the function $\underline{\underline{u}}$ satisfies the next relation with probability one

$$(6.3);\qquad \sum_{i=1}^{N} [\int_{R^1} dx \int_0^T \{ v_t^i u^i + (a^i v^i)_x u^i + (\sum_{j=1}^{N} A_{ji} v^j) u^i + B^i v^i \} dt$$

$$+ d_i \int_{R^1} dx \int_0^T v_x^i u^i dZ_t + \int_{R^1} v^i(0,x) u_0^i(x) dx \,] = 0.$$

6.2. Characteristic Solutions.

In the last section of Chapter V, we have remarked that the solution of (5.1)-(5.2), constructed by the characteristic method, satisfies the relation (5.33). Conversely we can use this relation to characterise a class of solutions whose expectations become solutions of corresponding parabolic equations.

Definition 6.2. A solution $\underline{\underline{u}}$ of the problem (6.1)-(6.2) is called characteristic, if it satisfies the following

$$(6.4); \qquad E[\int_{G[0,T]} \{ v \check{u}^i - d_i v_x u^i \} dt dx] = 0 \quad (i = 1,..,N)$$

where $\check{u}^i = \frac{\partial}{\partial Z_t} u^i$ and v is an arbitrary test function.

Now we have

Theorem 6.1; (i) The expectation $\underline{\underline{w}}(t,x) = E\,\underline{\underline{u}}(t,x,\omega)$ of a characteristic solution $\underline{\underline{w}}$ of (6.1)-(6.2) solves the following problem

$$(6.5); \quad \left\{ \begin{aligned} \frac{\partial}{\partial t} w^i + a^i(t,x) \frac{\partial}{\partial x} w^i = \frac{d_i^2}{2} \frac{\partial^2}{\partial x^2} w^i \\ + \sum_{j=1}^{N} A_{ij} w^j + B^i \end{aligned} \right.$$

$$\left| \; w^i(0,x) = u_0^i(x) \qquad (i = 1,..,N). \right.$$

(ii) Conversely if a solution $\underline{\underline{u}}$ of (6.1)-(6.2) is such that the expectation $\underline{\underline{w}} = E\underline{\underline{u}}$ solves the problem (6.5), then $\underline{\underline{u}}$ must be characteristic.

Proof. (i) Taking the expectation of both sides in (6.3), we find

$$(6.3)'; \qquad \sum_{i=1}^{N} \left[\int_{R^1} dx \int_0^T \left\{ v_t^i w^i + (a^i v^i)_x w^i + \left(\sum_{j=1}^{N} A_{ji} v^j \right) w^i + B^i v^i \right\} dt \right.$$

$$\left. + \frac{d_i}{2} \int_{R^1} dx \int_0^T v_x^i w^i dt + \int_{R^1} v^i(0,x) u_0^i(x) dx \right] = 0.$$

Since the function $\underline{\underline{u}}$ is characteristic, the relation above with the condition (6.4) yields the conclusion.

(ii) Comparing the equation in (6.5) with (6.3)', we get the conclusion. Q. E. D.

As for the existence of such solutions, we have

Theorem 6.2. If the functions $a^i(t,x)$, $A_{ij}(t,x)$, $B^i(t,x)$ and $u_0^i(x)(i,j=1,..,N)$ satisfy the same conditions stated in (A.1) and (A.2) of Theorem in Chapter V. Then there exist a characteristic solution of (6.1)-(6.2) and its expectation is the unique solution of (6.5).

The proof of this theorem is given in the same way as that of Theorem in Chapter V.

6. 3. Comments ; (i) Theorem 6.1 does not deny the existence of a solution of (6.1)-(6.2) which is not characteristic but solves the Cauchy problem of another system of parabolic equations. Therefore it is interesting and important to study the following

Problem; Does there exist a solution non-characteristic of the problem (6.1)-(6.2).

To study the problem above is to ask if the characteristic condition is independent of the system of equations (6.1). For the moment, the author does not have a good answer for it but he thinks that the existence of non-characteristic solutions is very little probable. However the answer may be, we can speak of the equivalence between two types of equations, limitting the class of stochastic solutions to be characteristic.

(ii) Because the white noise $\dot{Z}$ is a random distribution, it may be more convenient to treat brownian particle equations as equations of random distributions. In fact, the definition of solutions or the characteristic condition suggest a necessity of the study in this direction. However, one of the main difficulties of this idea is that we must treat the product of two random distributions like $u_x\dot{Z}$. The author is sure to overcome this point with the help of the theory of distributions and of the theory of B-derivatives.

CHAPTER VII. CONCLUSION

We have studied the problems P.1), P.2) and $P.3)_{(I)}$, $P.3)_{(II)}$ in this dissertation and obtained almost complete answers to them. In this chapter, we summarize the main results.

In Chapter II, we have considered the differentiabilities of stochastic processes with respect to the Brownian motion process, and established some properties about B-differentiable processes, such as the almost sure (or, stochastic) continuities of the processes. We have also considered the integral representations of B^+-differentiable processes and obtained a similar result to that obtained by E. Nelson. Our result states that the B^+-differentiation and the stochastic integration are the dual operations to each other; Given the M- and B^+-derivatives of a square integrable stochastic process X_t, we can reconstruct this process by the integrations of these derivatives.

In Chapyer III, we have introduced the new types of stochastic integrals which are connected with the Ito's integral,(the relation (3.2)). Among those integrals, we have investigated the properties of the integral $\mathcal{J}_{\frac{1}{2}}^{+}(f)$ in some details and found the relation between this integral and the limit of a sequence of random Stieltjes integrals. We have also refered to other types of stochastic integrals, namely the integral of Stratonovich and that of Stratonovich-Fisk, and we have compared them with our integral of index $\frac{1}{2}$ to find that these integrals coincide with each other for a class of random functions. Moreover we have found that our integral is applicable to such functions that do not admit the differential representation of Ito, while the other integrals are not so.

In Chapter IV, we have studied the initial value problem of the partial differential equation with the white noise as an external random force. The existence and the uniqueness of solutions were established, moreover the explicit form of this solution was obtained. Based on these results, we have considered the question on the validity of the hypothesis of the additive noise in those stochastic theories such as the theory of stochastic controls, the statistical theory of communications, etc.. As a result of considerations on the practical meaning of the unique solution, we have reached to give a precise definition of the additive noise and investigated its statistical properties. Applying the result to an example which corresponds to a simple situation in a communication system, we have found that there is a situation where we can estimate the variables of the transmission medium by the observation of the pure noise.

In Chapter V, we have studied the initial value problem of the partial differential equation including the white noise as its coefficient (the Brownian particle equation) and shown the existence of solutions by constructing one with the characteristic method. We have found that the average of this constructed solution satisfies a parabolic equation and that the solution represents a figure of a heat flow rather than that of a wave, while the solution of the previous problem (Chapter IV) still conserves the figure of a wave.

As a consequence, we have found that a parabolic equation has a diffusion process as its characteristic line.

In Chapter VI, we have given an answer to the question of uniqueness of solutions of a system of Brownian particle equations. Our results state that all the characteristic solutions of the problem have the same average which satifies a parabolic equation. In other words, we can assure the uniqueness of the average of solutions limitting ourselves to the class of characteristic solutions.

Throughout the discussions, we have restricted our attentions to the one-dimensional cases. The author thinks that the extension of discussions to multi-dimensional cases may not be carried in the direct ways. Moreover he thinks that the extensions to a more general case will offer us new problems and will produce fruitful results.

We have considered the problems in Chapters IV,V and VI as those of wave propagation in random media. However, as we have explained in the first chapter, the results obtained in this dissertation may be applicable to other physical problems. As for the applications of our theory, we have numerous ideas to imagine but not so much of results to express. Because the theory of stochastic partial differential equations with the white noise as their coefficients have been just settled at its starting point.

REFERENCES

[1] R. C. Bourret : *Stochastically perturbed fields, with applications to wave propagation in random media,* Nuovo Cimento, 26 (1962).

[2] S. Chandrasekhar : *Stochastic problems in physics and astronomy,* Reviews of Modern Physics, 15 (1943), 1 - 89.

[3] L. A. Chernov : *Wave propagation in random media,* McGraw-Hill, New York, 1960.

[4] E. D. Conway : *Generalized solutions of linear partial differential equations with discontinuous coefficients and the uniqueness question for multidimensional quasilinear conservation laws,* J. of Math. Anal and Appl., 18 (1967), 238-251.

[5] E. A. Coddington and N. Levinson : *Theory of Ordinary Differential Equations,* McGraw-Hill, New York, 1965.

[6] J. L. Doob : *Stochastic Processes,* Wiley, New York, 1953.

[7] A. Einstein : *Investigations of the Theory of the Brownian Movement,* edited with notes by R. Fürth, translated by A. D. Cowper, Dover, 1956.

[8] G. Ford, M. Kac and P. Mazur : *Statistical mechanics of assemblies of coupled oscillators,* J. Mathematical Phys., 6 (1965), 504-515.

[9] U. Frisch : *Propagation d'ondes dans un milieu aleatoire unidimensional,* Compt. Rend., 261 (1965), 55-57.

[10] U. Frisch : *Wave Propagation in Random Media,* in "*Probabilistic Methods in Applied Mathematics*" (Bharucha-Reid, ed.) vol. 1, Academic Press, New York, 1968.

[11] U. Frisch : *Wave propagation in random media (I): A theory of multiple scattering,* Institute d'Astrophysique, Paris, 1965.

[12] D. L. Fisk : *Quasi-Martingales,* Trans. Amer. Math. Soc., 120 (1965), 369-389.

[13] I. M. Gel'fand and N. Ya. Vilenkin : *Generalized Functions IV,* Academic Press, New York, 1964.

[14] K. Hasselmann : *Feynman diagrams and interaction rule of wave-wave scattering processes,* Rev. Geophys., 4 (1966), 1-32.

[15] D. Isaacson : *Stochastic integrals and derivatives,* The An Annals of Math. Statis., 40 (1969), 1610-1616.

[16] K. Ito : *Stochastic integral,* Proc. Imp. Acad. Tokyo, vol. 20 No. 8, (1944).

[17] K. Ito : *On a stochastic integral equation,* Proc. Imp. Acad. Tokyo, 22 (1946).

[18] K. Ito : *On a formula concerning stochastic differentials,* Nagoya Math. J., III, (1950).

[19] K. Ito : *Stationary random distributions,* Mem. of College of Sci., Univ. of Kyoto, Series A. vol. xxviii. Math., No 3, (1953).

[20] K. Ito : *Probability Theory (in Japanese)*, Iwanami-Shoten, Tokyo, 1963.

[21] K. Ito : *Lectures on Stochastic Processes*, Tata Institute, Bombay ; 1967.

[22] G. Kallianpur and C. Striebel : *Stochastic differential equations occurring in the estimation of continuous parameter stochastic processes*, Theory Probability Appl., 14 (1969), 567-594.

[23] G. Kallianpur and C. Striebel : *A stochastic differential equation of Fisk type for estimation and nonlinear filtering problems*, SIAM J. Appl., Math. 21 (1971).

[24] J. B. Keller : *Wave propagation in random media*, Proc. Symp. Appl. Math. 13, pp 227-246. Amer. Math. Soc., Providence, Rhode Island, 1962.

[25] J. B. Keller : *Stochastic equations and wave propagation in random media*, Proc. Symp. Appl. Math. 16, pp 145-170, Amer. Math. Soc., Providence, Rhode Island, 1964.

[26] H. Kushner : *Stochastic Stability and Controls*, Academic Press, New York, 1967.

[27] T. Nakamizo : *Considerations on modeling of stochastic processes in continuous-time control theory*, Seigyo-Kogaku, 14 (1970).

[28] E. Nelson : *Dynamical Theorie of Brownian Motion*, Princeton Univ. Press, Princeton, New Jersey, 1967.

[29] S. Ogawa : *On a Riemann definition of the stochastic integral (I),* Proc. Japan Acad. 46 (1970),

[30] S. Ogawa : *On a Riemann definition of the stochastic integral (II),* Proc. Japan Acad. 46 (1970),

[31] S. Ogawa : *A partial differential equation with the white noise as a coefficient,* Z. Wahrscheinlichkeitstheorie 28, 53-71 (1973),

[32] S. Ogawa : *Wave propagation in the presence of the white noise,* in preparation (1975),

[33] S. Ogawa : *Equation uni-dimensionelle de particule brownienne,* (in japanese) Master Thesis, Kyoto Univ. (1969),

[34] S. Ogawa : *Sur la question d'unicité des solutions du problèm de Cauchy d'un système d'équations de particule brownienne,* to appear in C.R.Acad. Paris (1975),

[35] H. Primas : *Über quantenmechanische System mit einem stochastichen Hamiltonoperator,* Helbv. Phys. Acta 34 (1961),

[36] S. Silver : *Delineation of problems and methodology in the subject of scattering by a statistically inhomogeneous medium,* in *"Electromagnetic Theory and Antennas",* (E.C. Jordan, ed.) pp 661-663, Pergamon Press, Oxford, (1963),

[37] R.L. Stratonovich : *Topics in the Theory of Random Noise,* vol. 1, Gorden and Breach, (Chapter 4) (1963),

[38] R.L. Stratonovich : *A new representation of stochastic integrals,* SIAM J., Control, 4 (1966).

[39] P.A. Sturrock : *Stochastic Acceleration,* Phys. Rev. 141 (1966).

[40] V.I. Tatarski : *Wave Propagation in a Turbulent Medium,* McGraw-Hill, New York, 1961.

[41] V.I.Tatarski and M.E.Gertsenshtein : *Propagation of Waves in a medium with strong fluctuations of the refractive index,* Zh. Eksperim. i Teor. Fiz. 44 (1963), 676-685.

[42] G.E.Uhlenbeck and L.S.Ornstein : *On the theory of Brownian motion,* Phys. Rev. 36 (1930), 823-841.

[43] E. Wong and M. Zakai : *On the relation between ordinary and stochastic differential equations,* Technical Report of Electronics Research Labo. Univ. of Calif. Berkley, Report No. 64-26, (1964).

[44] E. Wong and M. Zakai : *On the convergence of the solutions of differential equations involving Brownian motion,* Technical Report of Electronics Research Labo. Univ. of Calif. Berkley, Report No. 65-5, (1964).